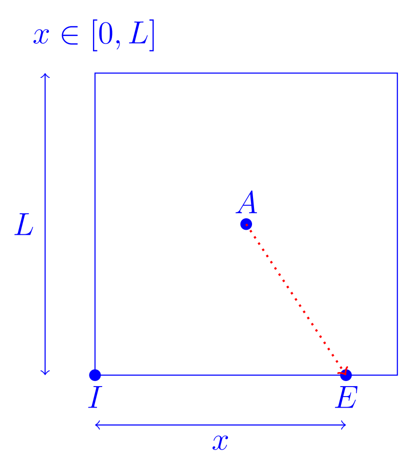
MAT 137Y: Calculus!
Problem Set 5
Due on Thursday, January 10 by 11:59pm via crowdmark

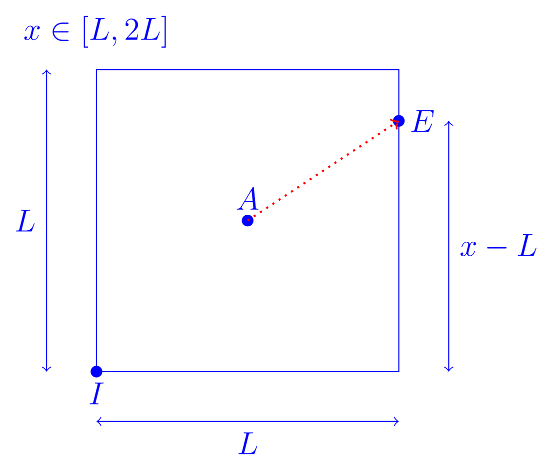
1. Alfonso is relaxing in the center of a square pool when suddenly he hears a yell. Ivan is standing at the corner of the pool, looking angry. Alfonso chooses a direction and starts swimming towards the side of the pool. Even though he does not know Ivan's exact speed, Alfonso knows he can outrun Ivan, so if he exits the pool before Ivan gets there, he is safe. Unfortunately, he is a very slow swimmer, and Ivan has started running around the edges of the pool towards Alfonso's exit point. Ivan is afraid of water and won't enter the pool. Also, once Alfonso chooses a direction, he never turns and he always remains straight, no matter how hard it may be.

   At what point should Alfonso try to exit the pool?

   Method 1:

   The following drawing describes the situation where I denotes the initial position of Ivan, A denotes the initial position of Alfonso, E denotes the exit point chosen by Alfonso, x denotes the distance travelled by Ivan to reach the point E and L denotes the side length of the pool.





   For $x \in [0, 2L]$, we denote by $D_A(x)$ the distance travelled by Alfonso to reach the point E. We want to minimize $\frac{D_A(x)}{x}$, or, equivalently since the square function is increasing on $[0, +\infty)$, $f(x) = \frac{D_A(x)^2}{x^2}$. The domain of study is $(0, 2L]$.
   Using the Pythagorean theorem, we obtain that

$$D_A(x)^2 = \begin{cases} x^2 - Lx + \frac{L^2}{2} & \text{if } x \in [0, L] \\ x^2 - 3Lx + \frac{5L^2}{2} & \text{if } x \in [L, 2L] \end{cases}$$

and therefore

$$f(x) = \begin{cases} 1 - \frac{L}{x} + \frac{L^2}{2x^2} & \text{if } x \in (0, L] \\ 1 - \frac{3L}{x} + \frac{5L^2}{2x^2} & \text{if } x \in [L, 2L] \end{cases}$$

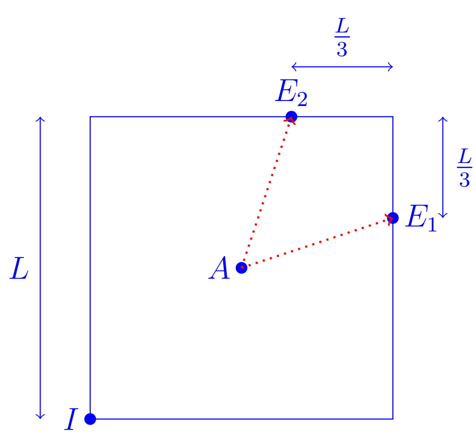
- Study on $(0, L]$.
  f is differentiable on $(0, L]$ and $f'(x) = \frac{L}{x^2}\left(1 - \frac{L}{x}\right)$.

x	0		L
$f'(x)$	‖	$-$	0
$f(x)$	‖ $+\infty$	↘	$\frac{1}{2}$

- Study on $[L, 2L]$.
  f is differentiable on $[L, 2L]$ and $f'(x) = \frac{L}{x^2}\left(3 - \frac{5L}{x}\right)$.

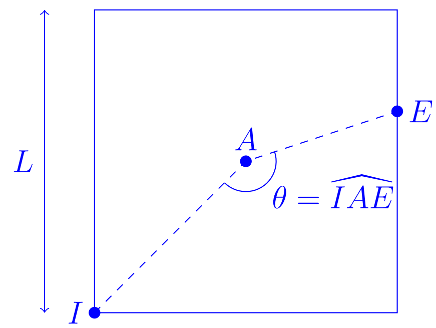
x	L		$\frac{5}{3}L$		$2L$
$f'(x)$		$-$	0	$+$	
$f(x)$	$\frac{1}{2}$	↘	$\frac{1}{10}$	↗	$\frac{1}{8}$

So the min of f on $(0, 2L]$ is reached at $x = \frac{5}{3}L$.
Hence Alfonso should exit the pool at a distance of $2L - \frac{5}{3}L = \frac{L}{3}$ from the corner opposite to the one where Ivan is initially located.



Method 2:

The following drawing describes the situation where I denotes the initial position of Ivan, A denotes the initial position of Alfonso, E denotes the exit point chosen by Alfonso and L denotes the side length of the pool.



By symmetry we may assume that $\theta \in [0, \pi]$.

Then, using trigonometry, we obtain that Ivan will travel the following distance to reach Alfonso's exit:

$$D_I(\theta) = \begin{cases} \left(\tan\left(\theta - \frac{\pi}{4}\right) + 1\right)\frac{L}{2} & \text{if } \theta \in \left[0, \frac{\pi}{2}\right] \\ \left(\tan\left(\theta - \frac{3\pi}{4}\right) + 3\right)\frac{L}{2} & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$

while Alfonso will travel the distance:

$$D_A(\theta) = \begin{cases} \frac{L}{2}\sec\left(\theta - \frac{\pi}{4}\right) & \text{if } \theta \in \left[0, \frac{\pi}{2}\right] \\ \frac{L}{2}\sec\left(\theta - \frac{3\pi}{4}\right) & \text{if } \theta \in \left[\frac{\pi}{2}, \pi\right] \end{cases}$$

We want to maximize the function $f(\theta) = \frac{D_I(\theta)}{D_A(\theta)}$ on $\theta \in [0, \pi]$.

(a) Study on $\left[0, \frac{\pi}{2}\right]$.
On this interval, we have

$$\begin{aligned} f(\theta) &= \sin\left(\theta - \frac{\pi}{4}\right) + \cos\left(\theta - \frac{\pi}{4}\right) \\ &= \sqrt{2}\sin(\theta) \end{aligned}$$

which is differentiable and

$$f'(\theta) = \sqrt{2}\cos(\theta)$$

The only critical point is $\theta = \frac{\pi}{2}$. Since f is continuous on the line segment $\left[0, \frac{\pi}{2}\right]$, it admits a max which is among the endpoints and the critical points.

- $f(0) = 0$
- $f\left(\frac{\pi}{2}\right) = \sqrt{2}$

(b) Study on $\left[\frac{\pi}{2}, \pi\right]$.
On this interval, we have

$$\begin{aligned} f(\theta) &= \sin\left(\theta - \frac{3\pi}{4}\right) + 3\cos\left(\theta - \frac{3\pi}{4}\right) \\ &= \sqrt{2}\sin(\theta) - 2\sqrt{2}\cos(\theta) \end{aligned}$$

which is differentiable and

$$f'(\theta) = \sqrt{2}\cos(\theta) + 2\sqrt{2}\sin(\theta)$$

The only critical point is $\theta = \pi - \arctan\left(\frac{1}{2}\right)$. Since f is continuous on the line segment $\left[\frac{\pi}{2}, \pi\right]$, it admits a max which is among the endpoints and the critical points.

- $f\left(\frac{\pi}{2}\right) = \sqrt{2}$
- $f\left(\pi - \arctan\left(\frac{1}{2}\right)\right) = \sqrt{10}$, indeed

$$\begin{aligned} f\left(\pi - \arctan\left(\frac{1}{2}\right)\right) &= \sqrt{2}\sin\left(\pi - \arctan\left(\frac{1}{2}\right)\right) - 2\sqrt{2}\cos\left(\pi - \arctan\left(\frac{1}{2}\right)\right) \\ &= \sqrt{2}\sin\left(\arctan\left(\frac{1}{2}\right)\right) + 2\sqrt{2}\cos\left(\arctan\left(\frac{1}{2}\right)\right) \\ &= \sqrt{2}\frac{1}{\sqrt{5}} + 2\sqrt{2}\frac{2}{\sqrt{5}} \\ &= \sqrt{10} \end{aligned}$$

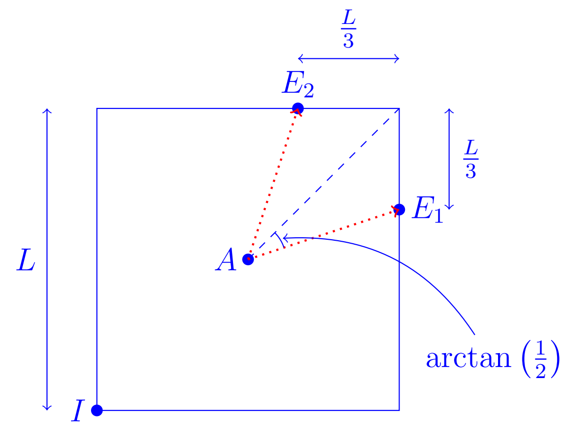
- $f(\pi) = \sqrt{8} = 2\sqrt{2}$

So the max of f on $[0, \pi]$ is reached at $\theta = \pi - \arctan\left(\frac{1}{2}\right)$.

Since

$$\begin{aligned} \tan\left(\frac{\pi}{4} - \arctan\left(\frac{1}{2}\right)\right) &= \frac{\sin\left(\frac{\pi}{4} - \arctan\left(\frac{1}{2}\right)\right)}{\cos\left(\frac{\pi}{4} - \arctan\left(\frac{1}{2}\right)\right)} \\ &= \frac{\cos\left(\arctan\left(\frac{\pi}{2}\right)\right) - \cos\left(\arctan\left(\frac{\pi}{2}\right)\right)}{\cos\left(\arctan\left(\frac{\pi}{2}\right)\right) + \cos\left(\arctan\left(\frac{\pi}{2}\right)\right)} \\ &= \frac{\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}}}{\frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}} \\ &= \frac{1}{3} \end{aligned}$$

Alfonso should exit the pool at a distance of $\frac{L}{2} - \frac{L}{2} \cdot \frac{1}{3} = \frac{L}{3}$ from the corner opposite to the one where Ivan is initially located.



2. Consider the following FALSE theorem and BAD proof.

False theorem
Let h be a function defined on an open interval I. Assume h is differentiable on I. Then h' is continuous on I.

Bad proof
Let $a \in I$. By definition, $h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$.
Since h is continuous, the limit of the numerator is 0. The limit of the denominator is also 0. Since h is differentiable, I can apply L'Hôpital's Rule.

$$h'(a) = \lim_{x \to a} \frac{h(x) - h(a)}{x - a} \overset{L'H}{=\!=} \lim_{x \to a} \frac{h'(x) - 0}{1 - 0} = \lim_{x \to a} h'(x).$$

I have proven that $h'(a) = \lim_{x \to a} h'(x)$. By definition, h' is continuous. $\square$

(a) Explain the error in the proof.

Recall the statement of l'Hôpital's rule:
IF

i. $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$,

ii. f and g are differentiable near a (except maybe at a),

iii. g and g' do not vanish near a (except maybe at a) and

iv. $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists or is $+\infty$ or is $-\infty$.

THEN

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

In the situation of the question, we do not know whether the assumption (iv) is satisfied or not, so we can't apply l'Hôpital's rule.

For instance, the function $h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$ is differentiable on $\mathbb{R}$ but $\lim_{x \to 0} h'(x)$ doesn't exist so we can't apply l'Hôpital's rule as in the *bad proof* for $a = 0$ (so the proof doesn't work).
Moreover, since h' is not continuous at 0, not only the proof doesn't work but the theorem is also false.

Interesting fact about the derivative: according to Darboux's theorem, if a function f is differentiable on an interval I then its derivative f' satisfies the conclusion of the IVT even if f' isn't continuous.

(b) "Fix" the theorem. (In other words, modify the statement of the theorem a little bit, either changing the hypotheses or the conclusion, so that it is true. There may be more than one way to do it.) You do not need to write the proof.

Let h be a function defined on an open interval I. Assume h is differentiable on I **and that for all** $a \in I$**,** $\lim_{x \to a} h'(x)$ **exists**. Then h' is continuous on I.

According to the above result, to check that a derivative is continuous, it is enough to check that it admits a limit at any point (then this limit is necessarily equal to the value of the derivative at this point: it's not necessary to check it).

<u>Proof:</u> Let $a \in I$. Let $f(x) = h(x) - h(a)$ and $g(x) = x - a$.
We know that

i. $\lim_{x \to a} f(x) = 0$ since h is continuous (since differentiable) and $\lim_{x \to a} g(x) = 0$,
ii. f and g are differentiable,
iii. $g(x) \neq 0$ for $x \neq a$ and $g'(x) \neq 0$,
iv. $\lim_{x \to a} \frac{f'(x)}{g'(x)} = \lim_{x \to a} h'(x)$ exists by assumption.

Hence, according to l'Hôpital's rule,

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a} = \lim_{x \to a} h'(x)$$

But, by definition of the derivative, since h is differentiable at a, we know that

$$\lim_{x \to a} \frac{h(x) - h(a)}{x - a} = h'(a)$$

We proved that $h'(a) = \lim_{x \to a} h'(x)$ so h' is continuous at a. ■

3. Let I be an open interval. Let $a \in I$. Let f be a function defined on I. Assume that f is continuous at a and that f is differentiable near a (except possibly at a). Consider the following two definitions:

- f *has a vertical tangent line at* a when $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \infty$ or $-\infty$.
- f *is funky at* a when $\lim_{x \to a} f'(x) = \infty$ or $-\infty$.

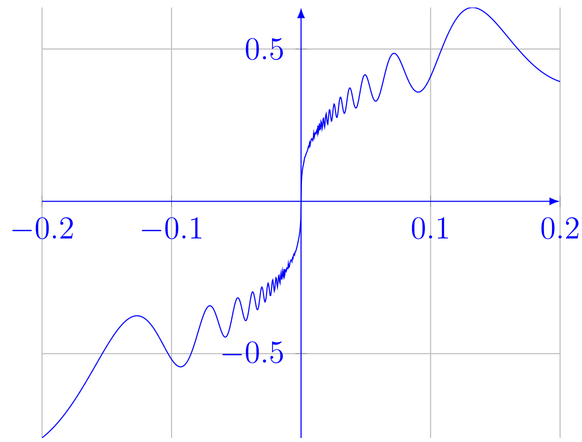
Is each one of the following statements true or false? If true, prove it. If false, construct a counterexample

(a) IF f is funky at a, THEN f has a vertical tangent line at a.
The above statement is true. Here is a proof:

- Assume that f is funky.
- Then
$$\lim_{x \to a} \frac{f'(x) - 0}{1 - 0} = \lim_{x \to a} f'(x) = +\infty \text{ or } -\infty$$
Hence we can apply l'Hôpital's rule:
$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \overset{L'H}{=\!=} \lim_{x \to a} \frac{f'(x) - 0}{1 - 0} = \lim_{x \to a} f'(x) = +\infty \text{ or } -\infty$$
- Hence f has a vertical tangent line at a.

■

(b) IF f has a vertical tangent line at a, THEN f is funky at a.
The above statement is false, here is a counter-example.
Let $f(x) = \begin{cases} \sqrt[3]{x} + x\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{otherwise} \end{cases}$, defined on $\mathbb{R} = (-\infty, +\infty)$.



Then,

- $\lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x\to 0} \dfrac{1}{x^{\frac{2}{3}}} + \sin\left(\dfrac{1}{x}\right) = +\infty$ so f has a vertical tangent line at 0.
- But f is not funky at 0 since $\lim\limits_{x\to 0} f'(x) \neq +\infty$ or $-\infty$.
  Indeed, f is differentiable on $\mathbb{R} \setminus \{0\}$ and
  $$f'(x) = \frac{1}{3x^{\frac{2}{3}}} + \sin\left(\frac{1}{x}\right) - \frac{\cos\left(\frac{1}{x}\right)}{x}$$
  For $u_n = \frac{1}{(2n+1)\pi}$ we have $u_n \to 0$ without stabilizing but
  $$\lim_{n\to\infty} f'(u_n) = \lim_{n\to\infty} \frac{((2n+1)\pi)^{\frac{2}{3}}}{3} + (2n+1)\pi = +\infty$$
  so
  $$\lim_{x\to 0} f'(x) \neq -\infty$$
  And for $v_n = \frac{1}{2n\pi}$ we have $v_n \to 0$ without stabilizing but
  $$\lim_{n\to\infty} f'(v_n) = \lim_{n\to\infty} \frac{(2n\pi)^{\frac{2}{3}}}{3} - 2n\pi = -\infty$$
  so
  $$\lim_{x\to 0} f'(x) \neq +\infty$$

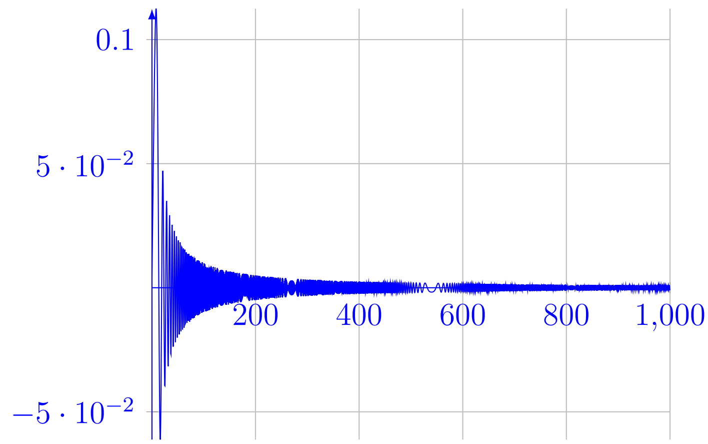
■

4. Let f be a function defined, at least, on an interval of the form (c, ∞) for some $c \in \mathbb{R}$. Assume f is differentiable.

Below are two claims. Are they true or false? If true, prove it. If false, provide a counterexample (and, as usual, show that your counterexample works).

(a) IF f has a horizontal asymptote as $x \to \infty$, THEN $\lim_{x\to\infty} f'(x) = 0$.
The above statement is false, here is a counter-example:

- The function $f(x) = \frac{\sin(x^2)}{x}$ is defined and differentiable on $(0, +\infty)$.



- The function f admits the horizontal asymptote $y = 0$ at $+\infty$ since

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \sin(x^2) \times \frac{1}{x} = 0$$

as the product of a bounded function by a function whose limit is 0.

- However $\lim_{x\to+\infty} f'(x)$ DNE (particularly, $\lim_{x\to+\infty} f'(x) \neq 0$).
Indeed, assume by contradiction that $\lim_{x\to+\infty} f'(x)$ exists.
Notice that $f'(x) = 2\cos(x^2) - \frac{\sin(x^2)}{x^2}$.
Since $\lim_{x\to+\infty} \dfrac{\sin(x^2)}{x^2} = 0$, we derive from the limit law for the addition that $2\cos(x^2) = f'(x) + \frac{\sin(x^2)}{x^2}$ has a limit at $+\infty$. Which is false.

(b) IF $\lim_{x \to \infty} f'(x) = 0$, THEN f has a horizontal asymptote as $x \to \infty$.

The above statement is false, here is a counter-example:

- The function $f(x) = \ln(x)$ is defined and differentiable on $(0, +\infty)$.
- We have
$$\lim_{x \to +\infty} f'(x) = \lim_{x \to +\infty} \frac{1}{x} = 0$$
- But
$$\lim_{x \to +\infty} f(x) = +\infty$$
so f doesn't admit a horizontal asymptote at $+\infty$.