MAT 137Y: Calculus!
Problem Set 4
Due on Thursday, November 22 by 11:59pm via crowdmark

Instructions:

- You will need to submit your solutions electronically. For instructions, see `http://uoft.me/CM137` . Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You will need to submit your answer to each question separately.
- This problem set is about implicit differentiation, inverse functions, and inverse trigonometric functions (Playlists 3 and 4).

1. Let $a, b > 0$. We want to study the curve with equation

$$(x^2 + y^2)^2 = ax^2 + by^2.$$

Notice that for each value of a and each value of b we get a different curve. You can see the graph on `http://tinyurl.com/mat137ps4` . (Strictly speaking we should also add the single point $(0, 0)$ to the graph you see at that url. For the purpose of this question, ignore the point $(0, 0)$.) You will find two sliders that allow you to change the values of a and b and see what happens to the graph.

(a) First, let's fix $a = 6$ and $b = 1$. Prove that the curve has exactly 6 points with a horizontal tangent line and find their coordinates.
Hint: Use implicit differentiation.

We are looking for points on the curve

$$(x^2 + y^2)^2 = 6x^2 + y^2 \tag{1}$$

such that $\frac{dy}{dx} = 0$ (we see y as a function of x locally around any point of the curve).
Hence we have to solve the following system (i.e. we are looking for points (x, y) satisfying simultaneously both following equations),

$$\begin{cases} (x^2 + y^2)^2 = 6x^2 + y^2 \\ \frac{dy}{dx} = 0 \end{cases}$$

The first equation ensures that we are looking at points on the curve, and the second that the tangent is horizontal at this point.
By differentiating (1) with respect to x, we obtain

$$
\begin{aligned}
& 2\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)(x^2+y^2)=12x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\\
\Leftrightarrow\quad & 4y(x^2+y^2)\frac{\mathrm{d}y}{\mathrm{d}x}-2y\frac{\mathrm{d}y}{\mathrm{d}x}=12x-4x(x^2+y^2)\\
\Leftrightarrow\quad & y\left(x^2+y^2-\frac{1}{2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=x(3-x^2-y^2)\\
\Leftrightarrow\quad & \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x(3-x^2-y^2)}{y\left(x^2+y^2-\frac{1}{2}\right)}
\end{aligned}
$$

Hence $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ if and only if $x(3-x^2-y^2)=0$ if and only if $x=0$ or $x^2+y^2=3$.

i. First case: Assume that $x=0$.
Hence, by substituting $x=0$ in (1), we obtain

$$
\begin{aligned}
& y^4=y^2\\
\Leftrightarrow\quad & y^2(y^2-1)=0\\
\Leftrightarrow\quad & y^2=1 \quad \text{(Indeed, we assumed that } (x,y)\neq(0,0) \text{ in the question)}\\
\Leftrightarrow\quad & y=-1 \text{ or } y=1
\end{aligned}
$$

Hence, we obtained the points $(0,-1)$ and $(0,1)$.

ii. Second case: Assume that $x^2+y^2=3$.
Then, by substituting $x^2+y^2=3$ in (1), we obtain:

$$
\begin{aligned}
& (x^2+y^2)^2=6x^2+y^2=5x^2+x^2+y^2\\
\Leftrightarrow\quad & 3^2=5x^2+3\\
\Leftrightarrow\quad & x^2=\frac{6}{5}
\end{aligned}
$$

By substituting the above in $x^2+y^2=3$, we get

$$y^2=3-\frac{6}{5}=\frac{9}{5}$$

Hence we obtained the four points $\left(\pm\sqrt{\frac{6}{5}},\pm\sqrt{\frac{9}{5}}\right)$

<u>Conclusion:</u> for $a=6$ and $b=1$, the curve admits exactly 6 points with a horizontal tangent line, which are

$$(0,-1),\ (0,1),\ \left(-\sqrt{\frac{6}{5}},-\sqrt{\frac{9}{5}}\right),\ \left(-\sqrt{\frac{6}{5}},\sqrt{\frac{9}{5}}\right),\ \left(\sqrt{\frac{6}{5}},-\sqrt{\frac{9}{5}}\right),\ \left(\sqrt{\frac{6}{5}},\sqrt{\frac{9}{5}}\right)$$

(b) Keep $a = 6$ and $b = 1$. Prove that the curve has exactly 2 points with a vertical tangent line and find their coordinates. Notice that it is not enough to find these points: we also want you to prove algebraically that it has no others.

Hint: Use implicit differentiation, thinking of x as a function of y.

Similarly to the above question, we are looking for points (x, y) such that

$$\begin{cases} (x^2+y^2)^2 = 6x^2+y^2 \\ \frac{\mathrm{d}x}{\mathrm{d}y} = 0 \end{cases}$$

By differentiating (1) with respect to y, we obtain

$$2\left(2x\frac{\mathrm{d}x}{\mathrm{d}y} + 2y\right)(x^2+y^2) = 12x\frac{\mathrm{d}x}{\mathrm{d}y} + 2y$$
$$\Leftrightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{y(\frac{1}{2} - (x^2+y^2))}{x(x^2+y^2-3)}$$

Hence $\frac{\mathrm{d}x}{\mathrm{d}y}$ vanishes if and only if $y = 0$ or $x^2 + y^2 = \frac{1}{2}$.

i. First case: $y = 0$.
By substituting $y = 0$ in (1), we obtain

$$x^4 = 6x^2$$
$$\Leftrightarrow \quad x^2(x^2-6) = 0$$
$$\Leftrightarrow \quad x^2 = 6 \quad \text{(Indeed, we assumed that } (x,y) \neq (0,0) \text{ in the question)}$$

So we obtained two points $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$.

ii. Second case: $x^2 + y^2 = \frac{1}{2}$.
We substitute $x^2 + y^2 = \frac{1}{2}$ in (1) and we obtain

$$(x^2+y^2)^2 = 6x^2+y^2 = 5x^2+x^2+y^2$$
$$\Leftrightarrow \quad \left(\frac{1}{2}\right)^2 = 5x^2 + \frac{1}{2}$$
$$\Leftrightarrow \quad x^2 = \frac{1}{20} - \frac{1}{10} = -\frac{1}{20} < 0$$

Since $x^2 \geq 0$, there is no point on the curve satisfying $x^2 + y^2 = \frac{1}{2}$.

Conclusion: for $a = 6$ and $b = 1$, the curve admits exactly 2 points with a vertical tangent line, which are $(-\sqrt{6}, 0)$ and $(\sqrt{6}, 0)$.

(c) Now play with the sliders and try different values of a and b. You will notice that sometimes the curve has exactly 2 points with a horizontal tangent line, and sometimes it has exactly 6 points with a horizontal tangent line. For which values of a and b does it have 6 and for which values does it have 2? Prove it.

Now, we want to find points (x, y) such that

$$\begin{cases} (x^2+y^2)^2 = ax^2+by^2 \\ \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \end{cases}$$

for arbitrary $a, b > 0$.

By differentiating

$$(x^2+y^2)^2 = ax^2+by^2 \tag{2}$$

with respect to x, we obtain:

$$2\left(2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}\right)(x^2+y^2) = 2ax+2by\frac{\mathrm{d}y}{\mathrm{d}x}$$

or equivalently,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x\left(\frac{a}{2}-(x^2+y^2)\right)}{y\left((x^2+y^2)-\frac{b}{2}\right)}$$

Hence, $\frac{\mathrm{d}y}{\mathrm{d}x}$ vanishes if and only if $x = 0$ or $x^2+y^2 = \frac{a}{2}$.

i. First case: $x = 0$.
By substituting $x = 0$ in (2), we obtain

$$\begin{aligned} & y^4 = by^2 \\ \Leftrightarrow\ & y^2(y^2-b) = 0 \\ \Leftrightarrow\ & y^2 = b \quad \text{(Indeed, we assumed that } (x,y) \neq (0,0) \text{ in the question)} \\ \Leftrightarrow\ & y = -\sqrt{b} \text{ or } y = \sqrt{b} \end{aligned}$$

Hence the points $(0, -\sqrt{b})$ and $(0, \sqrt{b})$ admit a horizontal tangent line.

ii. Second case: $x^2+y^2 = \frac{a}{2}$.
By substituting $x^2+y^2 = \frac{a}{2}$ in (2), we obtain

$$\begin{aligned} & (x^2+y^2)^2 = ax^2+by^2 \\ \Leftrightarrow\ & \left(\frac{a}{2}\right)^2 = a\left(\frac{a}{2}-y^2\right)+by^2 = \frac{a^2}{2}+(b-a)y^2 \\ \Leftrightarrow\ & a^2 = 4(a-b)y^2 \end{aligned}$$

If $a \leq b$ then $a^2 \leq 0$ which is not possible (a^2 can't vanish since $a > 0$).
Otherwise, if $a > b$, we have $y^2 = \frac{a^2}{4(a-b)} > 0$.
By substituting the above in $x^2 + y^2 = \frac{a}{2}$, we obtain

$$x^2 = \frac{a}{2} - \frac{a^2}{4(a-b)} = (a-2b)\frac{a}{4(a-b)}$$

If $a = 2b$, then $x = 0$ and this case has already been treated.
If $a < 2b$, then $x^2 < 0$ which is not possible, and, under this assumption this case doesn't produce new points with a horizontal tangent line.
If $a > 2b$, then we have 4 additional distinct points admitting a horizontal tangent line, whose coordinates are $\left(\pm\sqrt{\frac{a(a-2b)}{4(a-b)}}, \pm\frac{a}{2\sqrt{a-b}}\right)$.

Conclusion: the curve admits exactly 6 points with a horizontal line if $a > 2b$, and exactly 2 points with a horizontal line if $a \leq 2b$.

(d) ***[Do not submit.]*** Repeat Question 1c for points with a vertical tangent line, instead of a horizontal tangent line.

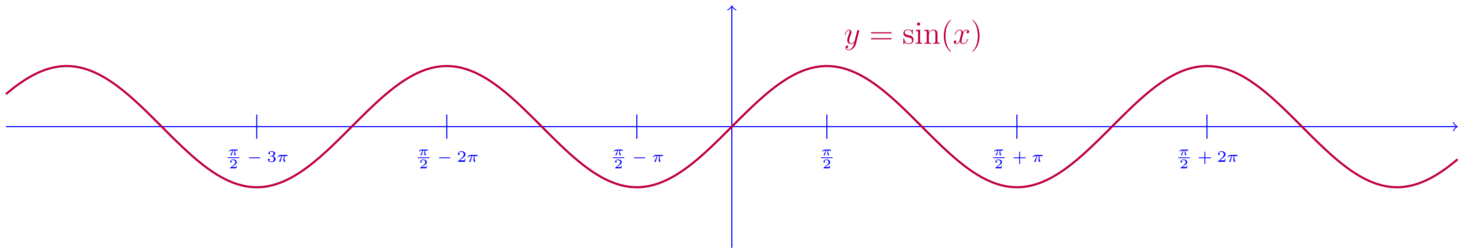
Hint: You can use a symmetry argument and your solution to 1c to answer this question without having to do any calculations.

Notice that if we set $\tilde{x} = y$ and $\tilde{y} = x$ then (2) becomes

$$(\tilde{x}^2 + \tilde{y}^2)^2 = b\tilde{x}^2 + a\tilde{y}^2$$

Hence, by symmetry with respect to the line $y = x$, the curve admits exactly 6 points with a vertical line if $b > 2a$, and exactly 2 points with a horizontal line if $b \leq 2a$.

2. Let us consider the function f defined by $f(x) = \sin x$. It is not one-to-one. For each $a \in \mathbb{R}$ we defined I_a to be the largest interval containing a such that the restriction of f to I_a is one-to-one.



(a) There are some values of $a \in \mathbb{R}$ for which the above definition does not make sense: the interval I_a is not well-defined. What are these values?

During this question, I will use several times the following fact:

> Let I be an interval. If there exists $n \in \mathbb{Z}$ such that $\frac{\pi}{2} + n\pi$ is an interior point of I, then sin isn't one-to-one on I. $(*)$

Indeed, if there exists $n \in \mathbb{Z}$ such that $\frac{\pi}{2} + n\pi$ is an interior point of I, then there exists $\varepsilon > 0$ such that $\frac{\pi}{2} + n\pi - \varepsilon$ and $\frac{\pi}{2} + n\pi + \varepsilon$ are both in I. But

$$\sin\left(\frac{\pi}{2} + n\pi + \varepsilon\right) = (-1)^n \cos(\varepsilon) = (-1)^n \cos(-\varepsilon) = \sin\left(\frac{\pi}{2} + n\pi - \varepsilon\right)$$

So sin isn't one-to-one on I. ■

Let $a \in \mathbb{R}$.
Assume that $a = \frac{\pi}{2} + n\pi$ for some $n \in \mathbb{Z}$.
According to $(*)$, a has to be an endpoint of I_a.
Notice that sin is one-to-one on $\left[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi\right]$ and on $\left[\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right]$ by (strict) monotoniticity of sin on these intervals.
Both of these intervals are of length π and can't be extended without loosing the injectiveness because of $(*)$.
Hence I_a is not uniquely defined in this case.

Assume that $a \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$.
Let $n = \left\lfloor \frac{a}{\pi} - \frac{1}{2} \right\rfloor$ and let $I_a = \left[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi\right]$.
Then a is an interior point of I_a, sin is one-to-one on I_a by strict monotoniticity and I_a can't be extended without loosing the injectiveness because of $(*)$.
Hence I_a is the largest interval containing a such that sin is one-to-one on I_a.

Conclusion: the interval I_a is well defined if and only if $a \in \mathbb{R} \setminus \left\{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\right\}$.
And the I_a are exactly the intervals $\left[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi\right]$ for $n \in \mathbb{Z}$.

(b) There may be different values of $a \in \mathbb{R}$ that produce the same interval I_a. What is the largest number of integers $a \in \mathbb{Z}$ such that they all produce the same interval I_a? (If you think the answer is n, you need to find n such integers as an example, and justify why it is impossible to find more than n.)

The largest number of integers in an interval I_a is **4**.
Let's prove it!.

- According to the above question $I_{11} = I_{12} = I_{13} = I_{14}$.
  Indeed, $\frac{\pi}{2} + 3\pi < 11$ and $14 < \frac{\pi}{2} + 4\pi$, so $\{11, 12, 13, 14\} \subset \left(\frac{\pi}{2} + 3\pi, \frac{\pi}{2} + 4\pi\right)$
  So we can find an interval I_a produced by at least 4 different integers.

- Assume by contradiction there exists an interval I_a produced by at least five distinct integers.
  Then I_a contains at least 5 consecutive integers and its length is at least 4. Which is not possible since, according to the previous question, the length of an interval I_a is exactly π.

Which ends the proof.

(c) Construct a set $A \subseteq \mathbb{R}$ such that any two different elements of A produce different intervals, and all possible interval I_a are produced by some element in A. We call this set a *"complete list of representatives"*.[1]

Let $A = \{k\pi,\ k \in \mathbb{Z}\}$.
Let's prove that A is a suitable complete list of representatives.

- Step 1: for any $a \in A$, the interval I_a is well defined.
  Let $a \in A$, then there exists $k \in \mathbb{Z}$ such that $a = k\pi$.
  Assume by contradiction, there exists $n \in \mathbb{Z}$ such that $a = \frac{\pi}{2} + n\pi$.
  Then

$$\begin{aligned} & k\pi = \frac{\pi}{2} + n\pi \\ \Leftrightarrow \quad & k = \frac{1}{2} + n \\ \Leftrightarrow \quad & \frac{1}{2} = k - n \end{aligned}$$

  So that $\frac{1}{2} \in \mathbb{Z}$, which is impossible.
  Hence, $A \subset \mathbb{R} \setminus \left\{\frac{\pi}{2} + n\pi,\ n\mathbb{Z}\right\}$.
  Therefore, according to question 1, I_a is well defined for $a \in A$.

- Step 2: two different elements of A produce different intervals.
  Assume there exist $k, k' \in \mathbb{Z}$ and $a \in \mathbb{R}$ such that $k\pi, k'\pi \in I_a$.
  We have $\sin(k\pi) = \sin(k'\pi) = 0$.
  Since sin is one-to-one on I_a, we obtain that $k\pi = k'\pi$ and $k = k'$.

- Step 3: any interval I_a is produced by an element of A.
  According to the first answer, the intervals I_a are exactly the intervals $\left[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi\right]$ for $n \in \mathbb{Z}$.
  Let fix such an interval $\left[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi\right]$ (or equivalently such an n).
  Notice that

$$\frac{\pi}{2} + n\pi = \left(n + \frac{1}{2}\right)\pi < (n+1)\pi < \left(n + \frac{3}{2}\right)\pi = \frac{\pi}{2} + (n+1)\pi$$

  So that $\left[\frac{\pi}{2} + n\pi, \frac{\pi}{2} + (n+1)\pi\right] = I_{(n+1)\pi}$ and $(n+1)\pi \in A$.

[1] A more formal way to say this is that $\forall b \in \mathbb{R}$ for which the interval I_b is well defined, there exists exactly one $a \in A$ such that $I_a = I_b$.

(d) For each $a \in A$, let us call $\heartsuit_a$ the inverse function of the restriction of f to I_a. What are the domain and the range of $\heartsuit_a$? Sketch its graph (labelling the axes properly).

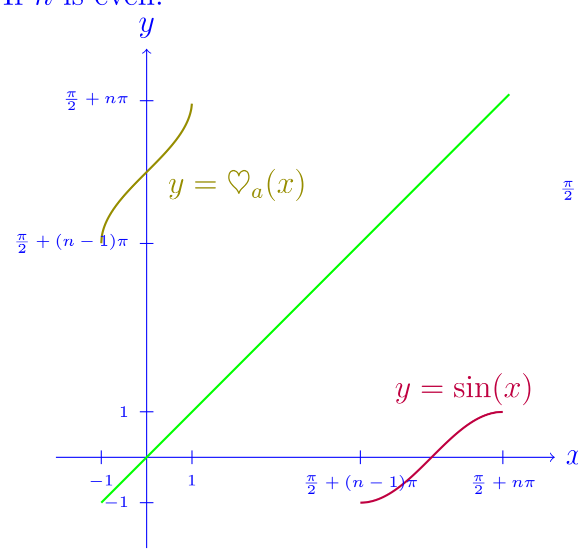
Let $a \in A$. Then there exists $n \in \mathbb{Z}$ such that $a = n\pi$.
According to the previous questions, $I_a = \left[\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right]$.

- We know that the restriction of sin to I_a is one-to-one.
- Moreover the range of this restriction is $[-1, 1]$.
  Indeed, let $c \in [-1, 1]$.
  We know that
  - sin is continuous on $I_a = \left[\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right]$,
  - if n is even then $\sin\left(\frac{\pi}{2} + (n-1)\pi\right) = -1$ and $\sin\left(\frac{\pi}{2} + n\pi\right) = 1$,
  - if n is odd then $\sin\left(\frac{\pi}{2} + (n-1)\pi\right) = 1$ and $\sin\left(\frac{\pi}{2} + n\pi\right) = -1$.

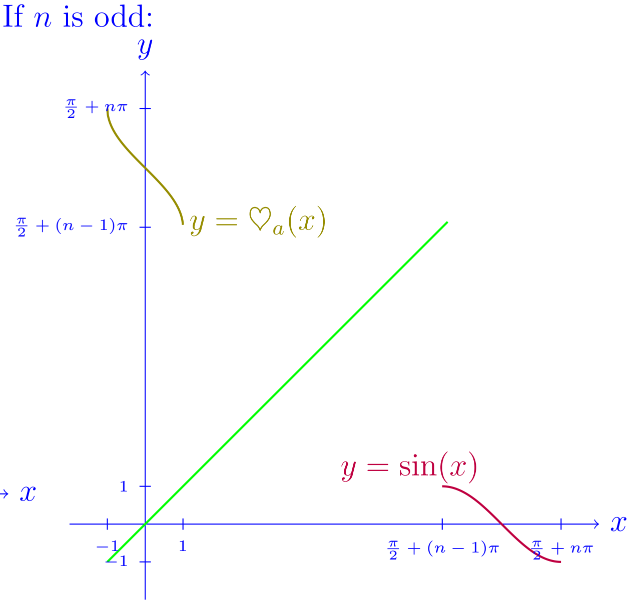
  Hence, according to the IVT, there exists $x \in I_a$ such that $\sin(x) = c$.

Hence the restriction of sin to I_a admits an inverse $\heartsuit_a$ whose domain is $[-1, 1]$ and whose range is I_a.

If n is even:



If n is odd:



The following observation, if proven, can simplify the answers of the following questions!

Assume that $a = n\pi$ for $n \in \mathbb{Z}$, then

$$\heartsuit_a(x) = (-1)^n \arcsin(x) + n\pi$$

Indeed, define $g : [-1, 1] \to I_a$ by $g(x) = (-1)^n \arcsin(x) + n\pi$.

Then $\forall x \in [-1, 1]$,

$$\begin{aligned}
\sin(g(x)) &= \sin((-1)^n \arcsin(x) + n\pi) \\
&= (-1)^n \sin((-1)^n \arcsin(x)) \\
&= (-1)^{2n} \sin(\arcsin(x)) \\
&= \sin(\arcsin(x)) \\
&= x
\end{aligned}$$

And, $\forall x \in I_a = \left[\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right]$,

$$\begin{aligned}
g(\sin(x)) &= (-1)^n \arcsin(\sin(x)) + n\pi \\
&= \arcsin((-1)^n \sin(x)) + n\pi \\
&= \arcsin(\sin(x - n\pi)) + n\pi \\
&= x - n\pi + n\pi \qquad \text{since } x - n\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \\
&= x
\end{aligned}$$

■

(e) For each $a \in A$, compute $\heartsuit_a(f(2018))$ and $f(\heartsuit_a(2018))$. Or, if they are not defined, explain why.

- Notice that $2018 \notin [-1,1]$ so $\forall a \in A$, $f(\heartsuit_a(2018))$ isn't define.

- Let $a \in A$. There exists $n \in \mathbb{Z}$ such that $a = n\pi$.
  We are going to prove that $\heartsuit_a(\sin(2018)) = (-1)^n(2018 - 642\pi) + n\pi$.

  - Method 1:
    Notice that $2018 - 642\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so that $2018 - 642\pi + n\pi \in I_a$.
    Then

$$\begin{aligned}\sin(2018) &= (-1)^{n-642}\sin(2018 - 642\pi + n\pi)\\ &= (-1)^n \sin(2018 - 642\pi + n\pi)\end{aligned}$$

    Hence, if n is even,

$$\heartsuit_a(\sin(2018)) = \heartsuit_a(\sin(2018 - 642\pi + n\pi)) = 2018 - 642\pi + n\pi$$

    Otherwise, if n is odd,

$$\begin{aligned}\heartsuit_a(\sin(2018)) &= \heartsuit_a(-\sin(2018 - 642\pi + n\pi))\\ &= \heartsuit_a(\sin(-2018 + 642\pi - n\pi))\\ &= \heartsuit_a(\sin(-2018 + 642\pi - n\pi + 2n\pi))\\ &= \heartsuit_a(\sin(-2018 + 642\pi + n\pi))\\ &= -(2018 - 642\pi) + n\pi\end{aligned}$$

    Indeed, $-2018 + 642\pi + n\pi \in I_a$ since $-2018 + 642\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

    Finally, we may conclude that if $a = n\pi$ then

$$\heartsuit_a(\sin(2018)) = (-1)^n(2018 - 642\pi) + n\pi$$

  - Method 2, using the above observation:

$$\begin{aligned}\heartsuit_a(\sin(2018)) &= (-1)^n \arcsin(\sin(2018)) + n\pi\\ &= (-1)^n(2018 - 642\pi) + n\pi\end{aligned}$$

(f) For each $a \in A$, derive an explicit formula for $\heartsuit_a'$.
Hint: Imitate the derivation in Video 4.7. Notice that it is not exactly the same.

Let $a \in A$. There exists $n \in \mathbb{Z}$ such that $a = n\pi$.

- Method 1:
  For all $x \in (-1, 1)$, we have $\sin(\heartsuit_a(x)) = x$.
  By differentiating with respect to x, we obtain:

  $$\begin{aligned} & \heartsuit_a'(x)\sin'(\heartsuit_a(x)) = 1 \\ \Leftrightarrow\ & \heartsuit_a'(x)\cos(\heartsuit_a(x)) = 1 \\ \Leftrightarrow\ & \heartsuit_a'(x) = \frac{1}{\cos(\heartsuit_a(x))} \end{aligned}$$

  Notice that we need to work with the open interval $(-1, 1)$ *since* $\cos(\heartsuit_a(x)) = 0$ *at* $-1, 1$.

  We derive from $\cos^2(\heartsuit_a(x)) + \sin^2(\heartsuit_a(x)) = 1$ that

  $$\cos(\heartsuit_a(x)) = \pm\sqrt{1 - \sin^2(\heartsuit_a(x))} = \pm\sqrt{1 - x^2}$$

  If n is even, then cos is positive on $\left(\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right)$ so that $\cos(\heartsuit_a(x)) = \sqrt{1 - x^2}$.

  Otherwise, if n is odd, then cos is negative on $\left(\frac{\pi}{2} + (n-1)\pi, \frac{\pi}{2} + n\pi\right)$ so that $\cos(\heartsuit_a(x)) = -\sqrt{1 - x^2}$.

  Finally, $\cos(\heartsuit_a(x)) = (-1)^n\sqrt{1 - x^2}$, and

  $$\heartsuit_a'(x) = \frac{(-1)^n}{\sqrt{1 - x^2}}$$

- Method 2, using the above observation:
  $\heartsuit_a'(x) = \frac{\mathrm{d}(-1)^n \arcsin(x) + n\pi}{\mathrm{d}x} = (-1)^n \arcsin'(x) = \frac{(-1)^n}{\sqrt{1-x^2}}$.