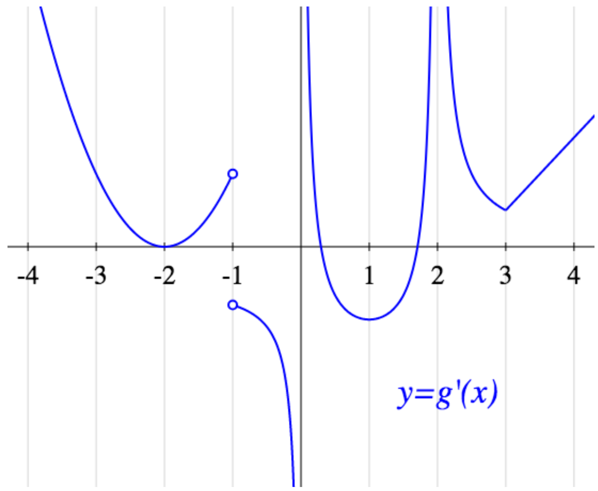
# MAT 137Y: Calculus!
## Problem Set 3
## Due on Thursday, November 1 by 11:59pm via crowdmark

1. We know the function $g$ has domain $\mathbb{R}$ and is continuous everywhere. We also know that $g(0) = 0$. Here is the graph of its derivative:




Sketch the graph of $g$.

Below is a sketch of the graph of $g$.

2. Consider the function $h$ given by the equation

$$h(x) = \sqrt{x + \sqrt{x + \sqrt{x + \ldots + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x+1}}}}}}}$$

where there are 2018 square roots in total. Find the equation of the line tangent to the graph of $h$ at the point with $x$–coordinate 0.

For $n \in \mathbb{N}_{>0}$, we define $h_n(x) = \sqrt{x + \sqrt{x + \sqrt{x + \ldots + \sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x+1}}}}}}}$ where there are $n$ square roots.

We are going to prove by induction that for any $n \in \mathbb{N}_{>0}$, $h_n'(0) = 1 - \dfrac{1}{2^n}$.

<u>Base case:</u> for $n = 1$.
Since $h_1(x) = \sqrt{x+1}$, we obtain that $h_1'(x) = \frac{1}{2\sqrt{1+x}}$ so that $h_1'(0) = \frac{1}{2}$.

<u>Induction step:</u> let $n \in \mathbb{N}_{>0}$.
Assume that $h_n'(0) = 1 - \dfrac{1}{2^n}$.
We want to show that $h_{n+1}'(0) = 1 - \dfrac{1}{2^{n+1}}$.
Since $h_{n+1}(x) = \sqrt{x + h_n(x)}$, we derive from the chain rule that $h_{n+1}'(x) = \dfrac{1 + h_n'(x)}{2\sqrt{x + h_n(x)}}$.
Hence $h_{n+1}'(0) = \dfrac{1 + h_n'(0)}{2\sqrt{0 + h_n(0)}} = \dfrac{1 + h_n'(0)}{2} = \dfrac{2 - \frac{1}{2^n}}{2} = \dfrac{2^{n+1} - 1}{2^{n+1}} = 1 - \dfrac{1}{2^{n+1}}$.
Which ends the induction proof.

According to the above formula, $h'(0) = 1 - \frac{1}{2^{2018}}$. Besides, $h(0) = 1$.
Hence the tangent to the graph of $h$ at $(0, 1)$ has the following equation:

$$y = \left(1 - \frac{1}{2^{2018}}\right) x + 1$$

3. The most common way to derive formulas for the derivatives of the six trig functions is the one you learned in the videos/class: we obtain the derivative of sin and cos from the definition ("the long way") and then we use the quotient rule to derive the rest. But we could have done it in other ways.

For the purpose of this problem, assume you know the basic differentiation rules (linearity, power, product, quotient, and chain) but that you do not know yet any of the formulas for derivatives of trig functions.

(a) Obtain a formula for the derivative of tan directly from the definition of derivative as a limit.

*Hint:* Write $\tan x = \dfrac{\sin x}{\cos x}$ and use the formulas for the sine of the sum and the cosine of the sum. This is similar to the derivation in Video 3.11.

First notice that for $a, b \in \mathbb{R}$ such that $\tan(a+b)$, $\tan(a)$ and $\tan(b)$ are well defined, we have

$$\begin{aligned}\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)}\\ &= \frac{\sin(a)\cos(b)+\cos(a)\sin(b)}{\cos(a)\cos(b)-\sin(a)\sin(b)}\\ &= \frac{\frac{\sin(a)}{\cos(a)}+\frac{\sin(b)}{\cos(b)}}{1-\frac{\sin(a)}{\cos(a)}\cdot\frac{\sin(b)}{\cos(b)}}\\ &= \frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}\end{aligned}$$

Then

$$\begin{aligned}\frac{\tan(x+h)-\tan(x)}{h} &= \frac{\frac{\tan(x)+\tan(h)}{1-\tan(x)\tan(h)}-\tan(x)}{h}\\ &= \frac{\tan(x)+\tan(h)-\tan(x)+\tan^2(x)\tan(h)}{h(1-\tan(x)\tan(h))}\\ &= \frac{\tan(h)+\tan^2(x)\tan(h)}{h(1-\tan(x)\tan(h))}\\ &= \frac{1}{1-\tan(x)\tan(h)}\left(\frac{\tan(h)}{h}+\tan^2(x)\cdot\frac{\tan(h)}{h}\right)\end{aligned}$$

Notice that the above is well defined as soon as $x \in \mathbb{R}\setminus\left\{\frac{\pi}{2}+n\pi,\ n\in\mathbb{Z}\right\}$ and $h$ is close to 0 but not 0.

Notice also that

$$\lim_{h\to 0}\frac{\tan(h)}{h} = \lim_{h\to 0}\left(\frac{1}{\cos(h)}\cdot\frac{\sin(h)}{h}\right) = \lim_{h\to 0}\frac{1}{\cos(h)}\cdot\lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

So that

$$\lim_{h\to 0}\frac{\tan(x+h)-\tan(x)}{h} = \lim_{h\to 0}\left(\frac{1}{1-\tan(x)\tan(h)}\left(\frac{\tan(h)}{h}+\tan^2(x)\cdot\frac{\tan(h)}{h}\right)\right)$$
$$= 1+\tan^2(x)$$

Hence tan is differentiable on $\mathbb{R}\setminus\left\{\frac{\pi}{2}+n\pi,\, n\in\mathbb{Z}\right\}$ and $\tan'(x)=1+\tan^2(x)$.

There is an alternative expression:

$$\tan'(x) = 1+\tan^2(x) = 1+\frac{\sin^2(x)}{\cos^2(x)} = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x).$$

(b) Use your answer to Question 3a and implicit differentiation on

$$\sec^2 x = 1+\tan^2 x$$

to obtain a formula for the derivative of sec.

By taking the derivative with respect to $x$ in the above identity, we obtain:

$$2\sec'(x)\sec(x) = 2\tan(x)\tan'(x) = 2\tan(x)\sec^2(x)$$

Furthermore, since $\sec(x)\neq 0$, we get

$$\sec'(x) = \tan(x)\sec(x).$$

(c) Use your answer to Question 3b to obtain a formula for the derivative of cos.

Since $\cos(x)=\frac{1}{\sec(x)}$, we obtain

$$\begin{aligned}\cos'(x) &= -\frac{\sec'(x)}{\sec^2(x)}\\ &= -\frac{\tan(x)\sec(x)}{\sec^2(x)}\\ &= -\frac{\tan(x)}{\sec(x)}\\ &= -\tan(x)\cos(x)\\ &= -\sin(x)\end{aligned}$$

We got

$$\cos'(x) = -\sin(x)$$

*Comment: the above proof works on* $\mathbb{R}\setminus\left\{\frac{\pi}{2}+n\pi,\, n\in\mathbb{Z}\right\}$*, however we can check that the formula is true on* $\mathbb{R}$*.*

(d) Use your answer to Question 3c and the equations

$$\cos x = \sin\left(\frac{\pi}{2} - x\right), \qquad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

to obtain a formula for the derivative of sin.

We derive from the identity $\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$ and the chain rule that

$$\sin'(x) = -\cos'\left(\frac{\pi}{2} - x\right)$$

Hence, using Question 3c, we obtain

$$\sin'(x) = \sin\left(\frac{\pi}{2} - x\right)$$

Finally,

$$\sin'(x) = \cos(x)$$

4. Let $f$ be a continuous function with domain $\mathbb{R}$. Assume that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$. Prove that $f$ takes all possible real values. In other words, prove that for every $y \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $f(x) = y$.

*Hint:* As part of your proof, you will need to use the IVT, the definition of $\lim_{x\to\infty} f(x) = \infty$, and the definition of $\lim_{x\to-\infty} f(x) = -\infty$. If you do not use the three of them (or something related), your proof is probably wrong.

WTS: $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, f(x) = y$.

- Let $y \in \mathbb{R}$.
- Since $\lim_{x\to-\infty} f(x) = -\infty$, there exists $M < 0$ such that
$$\forall x \in \mathbb{R}, (x < M \implies f(x) < y) \tag{1}$$
- Since $\lim_{x\to+\infty} f(x) = +\infty$, there exists $N > 0$ such that
$$\forall x \in \mathbb{R}, (x > N \implies f(x) > y) \tag{2}$$
- We know that
  (i) $f$ is continuous on $[M-1, N+1]$,
  (ii) $f(M-1) < y$ by (1), and
  (iii) $f(N+1) > y$ by (2).
  Hence, according to the IVT, there exists $x \in [M-1, N+1] \subset \mathbb{R}$ such that $f(x) = y$.