MAT 137Y: Calculus!
Problem Set 2
Due on Thursday, October 11 by 11:59pm via crowdmark

Instructions:

- You will need to submit your solutions electronically. For instructions, see `http://uoft.me/CM137` . Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You will need to submit your answer to each question separately.
- This problem set is about the definition of limit and basic properties of limits (Playlist 2 up to Video 2.13)

1. (*Note:* Before you attempt this problem, solve Problem 2.1-7 in the textbook. Otherwise you will find this question too difficult.)

   Sketch the graph of a function f that satisfies all 12 conditions below simultaneously. For this question only, you do not need to prove or explain your answer, as long as the graph is correct and very clear. If you cannot satisfy all the properties at once, get as many as you can.

   (a) The domain of f is, at least, $(-5, 5)$

   (b) $\lim_{x\to a} f(x)$ exists for every a in the domain of f, except $a = 0$.

   (c) $\lim_{x\to 0} f(x)$ DNE

   (d) $\lim_{x\to 1} f(x) = 0$

   (e) $\lim_{x\to -1} f(x) = 0$

   (f) $\lim_{x\to 3} f(x) = 0$

   (g) $\lim_{x\to -3} f(x) = 0$

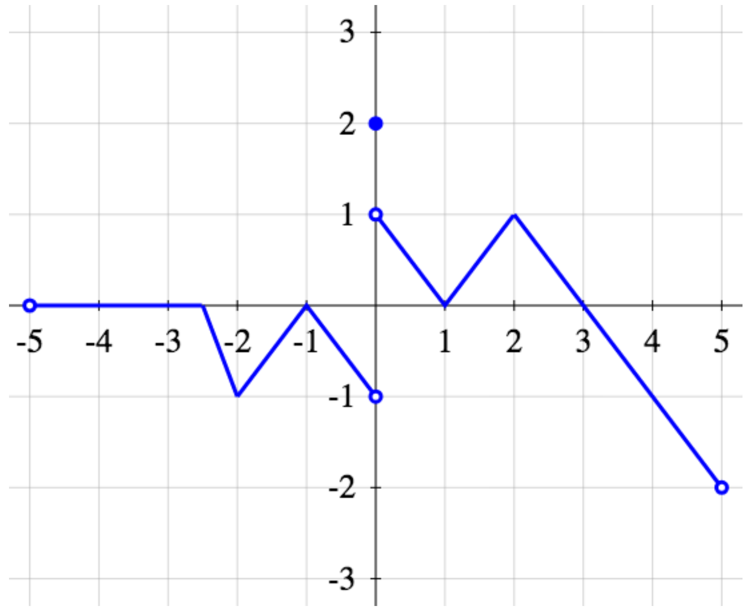
   (h) $\lim_{x\to 0} f(f(x)) = 0$

   (i) $\lim_{x\to 1} f(f(x)) = 1$

   (j) $\lim_{x\to -1} f(f(x)) = -1$

   (k) $\lim_{x\to 3} f(f(x))$ DNE

   (l) $\lim_{x\to -3} f(f(x)) = 2$



2. Let $a \in \mathbb{R}$. Let f and g be functions defined at least on an interval centered at a, except possibly at a. Is each of the following claims true or false? If it is false, show it with a counterexample. If it is true, prove it. (The proof should be a short, "one-line" proof using the properties of limits you already know. Do not use the formal definition of limit. No epsilons allowed in this question.)

   (a) IF $\lim_{x\to a} [f(x) + g(x)]$ exists and $\lim_{x\to a} f(x)$ exists, THEN $\lim_{x\to a} g(x)$ exists.
   This statement is TRUE. Let's prove it.
   Since $\lim_{x\to a} [f(x) + g(x)]$ and $\lim_{x\to a} f(x)$ both exist, the limit law for the difference tells us that $g(x) = \big(f(x) + g(x)\big) - f(x)$ admits a limit when x tends to a.

   (b) IF $\lim_{x\to a} [f(x) \cdot g(x)]$ exists and $\lim_{x\to a} f(x)$ exists, THEN $\lim_{x\to a} g(x)$ exists.
   This statement is FALSE. Below is a counter-example.

   - Take $f(x) = x^2$ and $g(x) = \frac{1}{x}$ defined on $(-\infty, 0) \cup (0, +\infty)$.
   - Then $\lim_{x\to 0} f(x) \cdot g(x) = \lim_{x\to 0} x = 0$ and $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$,
   - But $\lim_{x\to 0} g(x) = \lim_{x\to 0} \dfrac{1}{x}$ doesn't exist.

3. Prove, directly from the formal definition of limit, that

$$\lim_{x\to 1}\left(x^3+2x\right)=3.$$

Write a proof directly from the definition. Do not use any of the limit laws.

WTS: $\forall \varepsilon>0,\ \exists \delta>0,\ \forall x\in\mathbb{R},\ \big(0<|x-1|<\delta \implies |x^3+2x-3|<\varepsilon\big)$

We present two different proofs (among many) with different suitable δ.

Proof 1:

- Let $\varepsilon>0$.
- Take $\delta=\min\left(\sqrt[3]{\frac{\varepsilon}{3}},\frac{\sqrt{\varepsilon}}{3},\frac{\varepsilon}{15}\right)$.
- Let $x\in\mathbb{R}$.
- Assume $0<|x-1|<\delta$.
- Then,

$$\begin{aligned}
\left|x^3+2x-3\right| &= \left|(x-1)(x^2+x+3)\right| \\
&= \left|(x-1)((x-1)^2+3x+2)\right| \\
&= \left|(x-1)((x-1)^2+3(x-1)+5)\right| \\
&= \left|(x-1)^3+3(x-1)^2+5(x-1)\right| \\
&\le |x-1|^3+3|x-1|^2+5|x-1| \text{ by the Triangle Inequality} \\
&< \delta^3+3\delta^2+5\delta \\
&\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

Hence $|x^3+2x-3|<\varepsilon$.

$\blacksquare$

Proof 2:

- Let $\varepsilon > 0$.
- Take $\delta = \min\left(1, \frac{\varepsilon}{9}\right)$.
- Let $x \in \mathbb{R}$.
- Assume $0 < |x-1| < \delta$.
- Notice that

$$\begin{aligned} |x-1| < \delta \leq 1 &\implies -1 < x-1 < 1 \\ &\implies 0 < x < 2 \\ &\implies \begin{cases} 0 < x^2 < 4 \\ 0 < x < 2 \end{cases} \\ &\implies 0 < x^2 + x + 3 < 9 \\ &\implies |x^2 + x + 3| < 9 \end{aligned}$$

and that $|x-1| < \delta \leq \frac{\varepsilon}{9}$.
Thus

$$\begin{aligned} |x^3 + 2x - 3| &= |(x-1)(x^2+x+3)| \\ &= |x-1| \cdot |x^2+x+3| \\ &< 9 \cdot \frac{\varepsilon}{9} \\ &= \varepsilon \end{aligned}$$

Hence $|x^3 + 2x - 3| < \varepsilon$.

$\blacksquare$

4. Let f be a function with domain $(-\infty, 0) \cup (0, \infty)$. Prove that

   IF $\lim_{x \to 0} f(x) = \infty$
   THEN $\lim_{x \to 0} f(x)$ does not exist

   Notes: Before you write this proof, make sure you understand the precise definition of "the limit is ∞" and the definition of "the limit does not exist". Notice that the definition of "the limit does not exist" is *not* the negation of "the limit is L". If your definitions are not correct, then your proof cannot possibly be correct, and you won't get any credit. Make sure to write a formal proof directly from the formal definitions, without using any limit laws or similar properties.

Remember that:

(a) "$\lim_{x \to 0} f(x) = \infty$" means:

$$\forall M \in \mathbb{R},\ \exists \delta > 0,\ \forall x \in (-\infty, 0) \cup (0, +\infty),\ \big(0 < |x| < \delta \implies f(x) > M\big)$$

(b) "$\lim_{x \to 0} f(x)$ doesn't exist" means:

$$\forall L \in \mathbb{R},\ \exists \varepsilon > 0,\ \forall \delta > 0,\ \exists x \in (-\infty, 0) \cup (0, +\infty),\ \big(0 < |x| < \delta \text{ and } |f(x) - L| \geq \varepsilon\big)$$

Before starting the proof, remember also that

$$\forall x, y \in \mathbb{R},\ |x - y| \geq |x| - |y| \tag{1}$$

Indeed, $|x| = |(x - y) + y| \leq |x - y| + |y|$ by the Triangle Inequality.

Proof of the statement:
We assume that $\lim_{x \to 0} f(x) = \infty$ and we want to show that $\lim_{x \to 0} f(x)$ doesn't exist.

- Let $L \in \mathbb{R}$.
- Take $\varepsilon = 1$, notice that $\varepsilon > 0$.
- Let $\delta > 0$.
- Since $\lim_{x \to 0} f(x) = \infty$, there exists $\delta_1 > 0$ such that for any $x \in (-\infty, 0) \cup (0, +\infty)$,

$$\text{if } 0 < |x| < \delta_1 \text{ then } f(x) > 1 + |L|. \tag{2}$$

- Take $x = \min\left(\frac{\delta}{2}, \frac{\delta_1}{2}\right)$, notice that $x \in (0, +\infty) \subset (-\infty, 0) \cup (0, +\infty)$.
- Then $0 < |x| = x \leq \frac{\delta}{2} < \delta$,
- And also,

$$\begin{aligned}
|f(x) - L| &\geq |f(x)| - |L| && \text{by (1)} \\
&\geq f(x) - |L| \\
&> 1 + |L| - |L| && \text{by (2) since } 0 < |x| = x \leq \tfrac{\delta_1}{2} < \delta_1 \\
&= 1 \\
&= \varepsilon
\end{aligned}$$

- We have well $0 < |x| < \delta$ and $|f(x) - L| \geq \varepsilon$.

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