MAT 137Y: Calculus!
Problem Set 1
Due on Thursday, September 27 by 11:59pm via crowdmark

Instructions:

- You will need to submit your solutions electronically. For instructions, see `http://uoft.me/CM137` . Make sure you understand how to submit and that you try the system ahead of time. If you leave it for the last minute and you run into technical problems, you will be late. There are no extensions for any reason.
- You will need to submit your answer to each question separately.
- This problem set is about the introduction to logic, notation, quantifiers, conditionals, definitions, and proofs (Playlist 1).

0. Read "Notes on collaboration" on the course website: `http://uoft.me/colaboration`

   Copy out the following sentence and sign below it, to certify that you have read the "Notes on Collaboration".

   > "I have read and understood the notes on collaboration for this course, as explained in the course website."

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1. Negate the following statement without using any negative words ("no", "not", "none", "zero", etc.):

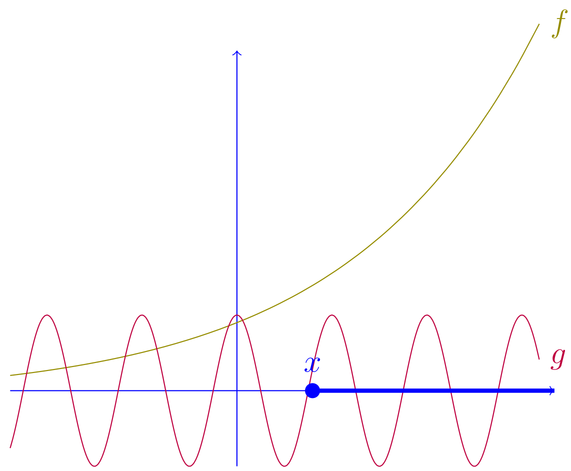
   > "A professor at a university in Ontario has written a book that has the property that every word in every even-numbered page begins with a letter that comes alphabetically before the letter it ends with."

   Every books written by any professor in any university in Ontario satisfy the property that there is an even-numbered page containing a word whose first letter is its last letter or a letter that comes after it in the alphabet.

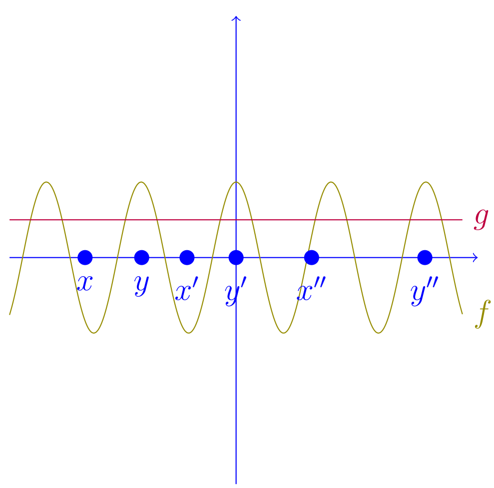
2. In this problem we will only consider (real-valued) functions with domain $\mathbb{R}$. We define two new concepts. Let f and g be two functions.

   - We say that f *is happier than* g when $\exists x \in \mathbb{R}$ s.t. $\forall y \in \mathbb{R}$,
   $$x < y \implies f(y) > g(y).$$
   - We say that f *is luckier than* g when $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$ s.t.
   $$x < y \ \textbf{AND} \ f(y) > g(y).$$

   Geometrically, f is happier than g if there exists a semiline $(x, +\infty)$ where the graph of f is above the graph of g, as in the example below.



   And f is luckier than g, if for every x there exists at least one point y after x where the graph f is above the graph of g, as illustrated below.



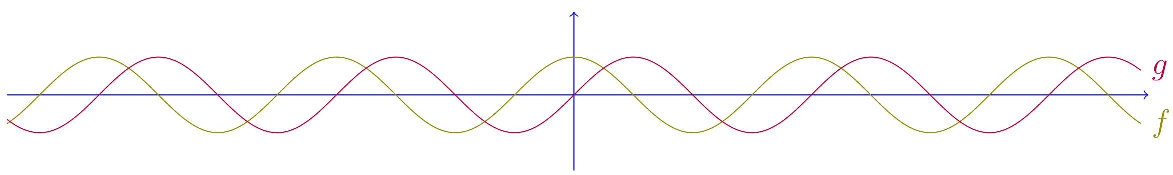
We also need a new piece of notation. Given any function f and any $t \in \mathbb{R}$, we define a new function, called f_t, via the equation

$$f_t(x) = f(x) + t$$

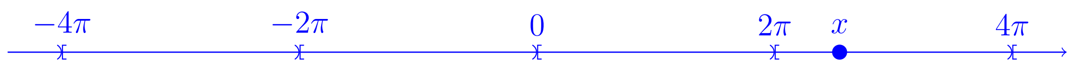
for all $x \in \mathbb{R}$.

Below are three claims. Which ones are true and which ones are false? If a claim is true, prove it. If a claim is false, show it with a counterexample.

(a) If f and g are two functions and f is luckier than g, then f is happier than g.
The statement is false. Indeed, below is a counter-example.
For $f(x) = \cos(x)$ and $g(x) = \sin(x)$, f is luckier than g but f is not happier than g.

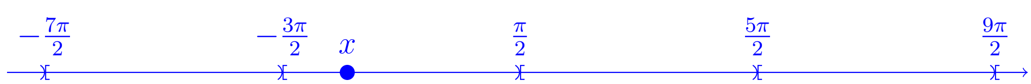


Let's prove that f is luckier than g. Let $x \in \mathbb{R}$.
There is a unique $k \in \mathbb{Z}$ such that $x \in [2k\pi, 2(k+1)\pi)$.



Take $y = 2(k+1)\pi$. Then $y > x$ and $f(y) = 1 > 0 = g(y)$.

Let's prove that f is not happier than g. Let $x \in \mathbb{R}$.
There is a unique $k \in \mathbb{Z}$ such that $x \in \left[\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2(k+1)\pi\right)$.



Then $y = \frac{\pi}{2} + 2(k+1)\pi$ satisfies $y > x$ and $f(y) = 0 \leq 1 = g(y)$.

(b) If f and g are two functions and f is happier than g, then f is luckier than g.
The statement is true. Let's prove it.
Let f and g be two functions. Assume that f is happier than g.
We want to show that f is luckier than g.
Let $x \in \mathbb{R}$.
Since f is happier than g, there exists $x' \in \mathbb{R}$ such that for any $y \in \mathbb{R}$, if $y > x'$ then $f(y) > g(y)$.
Take $y = \max(x+1, x'+1)$.
Then $y > x$ and $f(y) > g(y)$ (since $y > x'$).

(c) For every function f there exists a function g such that for every $t \in \mathbb{R}$, g is happier than f_t.
The statement is true. Let's prove it.
Let f be a function with domain $\mathbb{R}$.
Fix $g(x) = f(x) + x$.
Let $t \in \mathbb{R}$. We want to show that g is happier than $f_t(x) = f(x) + t$.
Take $x = t$. Let $y \in \mathbb{R}$.
If $y > x$ then $g(y) = f(y) + y > f(y) + x = f(y) + t = f_t(y)$.

3. Prove by induction that for every positive integer n, the number $2^{3n} + 6$ is a multiple of 7.
Base case: for $n = 1$.
$2^{3 \cdot 1} + 6 = 2^3 + 6 = 8 + 6 = 14 = 2 \cdot 7$.
Inductive step: let n be a positive integer.
Assume that $2^{3n} + 6$ is a multiple of 7, i.e. there exists $k \in \mathbb{N}$ such that $2^{3n} + 6 = 7k$.
We want to show that $2^{3(n+1)} + 6$ is a multiple of 7.

$$\begin{aligned}
2^{3(n+1)} + 6 &= 2^{3n+3} + 6 \\
&= 2^3 2^{3n} + 6 \\
&= 8 \cdot 2^{3n} + 6 \\
&= 8(2^{3n} + 6 - 6) + 6 \\
&= 8(2^{3n} + 6) - 8 \cdot 6 + 6 \\
&= 8(2^{3n} + 6) - 7 \cdot 6 \\
&= 8 \cdot 7 \cdot k - 7 \cdot 6 \qquad \textit{(Inductive hypothesis)} \\
&= 7(8k - 6)
\end{aligned}$$