MAT335H1F Lec0101 Burbulla

Chapter 5 and 6 Lecture Notes

Fall 2012

Calculus Review: The Intermediate Value Theorem

If a function $f(x)$ is continuous on the closed interval $[a, b]$ then it has no discontinuities; that is, the graph of the function has no holes, jumps or vertical asymptotes. Putting it another way, the graph of $y = f(x)$ must be a single curve, with no gaps in it. To draw it, your pencil would never leave the paper. The Intermediate Value Theorem is a mathematical restatement of this idea:

> *If f is a continuous function on the closed interval $[a, b]$ and K is any value between $f(a)$ and $f(b)$, then there is a number c in the interval (a, b) such that $f(c) = K$.*

K is called an intermediate value. IVT says that every continuous function on a closed interval must pass every intermediate value.

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Chapter 6: Bifurcations
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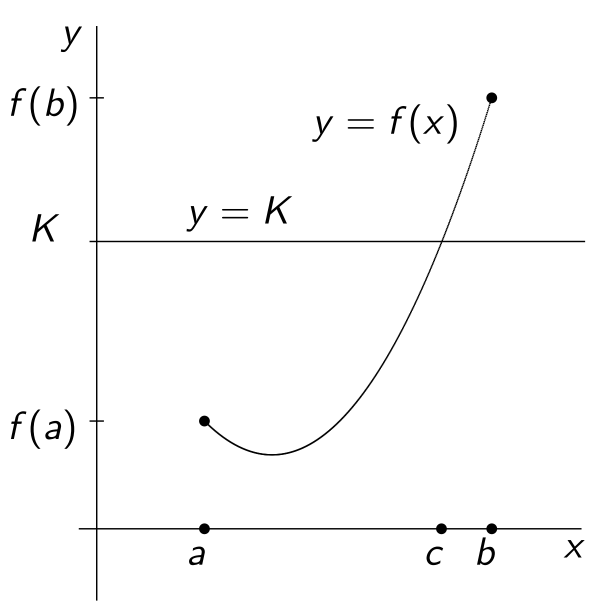
Picture for IVT



- If f is continuous on the closed interval $[a, b]$
- and
$$f(a) \leq K \leq f(b),$$
or
$$f(a) \geq K \geq f(b),$$
- then there is a number c in the interval $[a, b]$ such that
$$f(c) = K.$$

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The Fixed Point Theorem

If $F : [a, b] \longrightarrow [a, b]$ is a continuous function on the closed interval $[a, b]$ then F has at least one fixed point in the interval $[a, b]$.

Proof: Let $h(x) = F(x) - x$; then h is a continuous function on $[a, b]$ and

$$h(a) = F(a) - a \geq 0 \text{ because } F(a) \geq a;$$

and

$$h(b) = F(b) - b \leq 0 \text{ because } F(b) \leq b.$$

By the Intermediate Value Theorem applied to h on $[a, b]$, there is a number $c \in [a, b]$ such that

$$h(c) = 0 \Leftrightarrow F(c) = c.$$

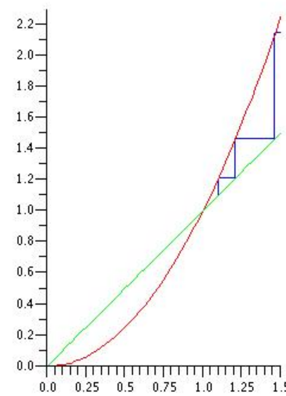
Remarks About the Fixed Point Theorem

1. The theorem states F has at least one fixed point in $[a, b]$; there could be more than one.
2. The Fixed Point Theorem only applies to a continuous function which maps the interval $[a, b]$ into itself. One frequent detail we will have to check in this course is if a function's range is contained in its domain.
3. The Fixed Point Theorem assumes F is continuous on a closed interval; the result may or may not hold true if the domain of F is not a closed interval.
4. The Fixed Point Theorem is an existence theorem; it doesn't indicate what the fixed point is.

Two Types of Fixed Points

The function $F(x) = x^2$ has two fixed points, $x = 0$ and $x = 1$. Orbits of x_0 under F behave markedly differently depending on which point x_0 is close to.

If $x_0 > 1$ then the orbit of x_0 under F tends to infinity.



Attracting and Repelling Fixed Points

On the other hand if $|x_0| < 1$ then orbits of x_0 under F tend to 0.

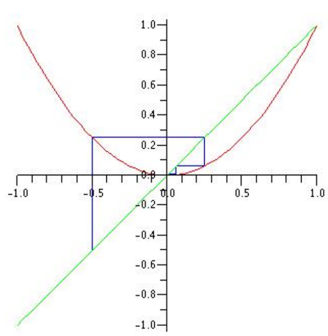


Figure: $-1 < x_0 < 0$

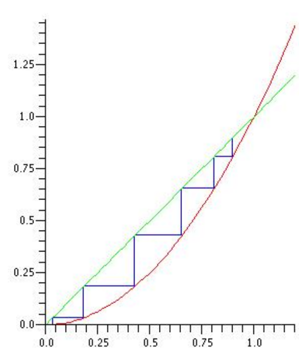


Figure: $0 < x_0 < 1$

We call 0 an attracting fixed point; we call 1 a repelling fixed point.

Examples

Let $L(x) = mx$. The only fixed point of L is $x = 0$. Whether or not $x = 0$ is an attracting fixed point or a repelling fixed point depends on m, the slope of L. Here are four examples:

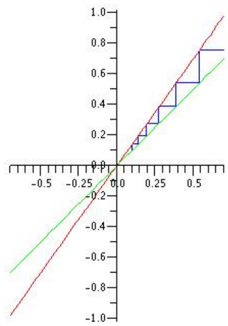


Figure: $m = 1.4; x_0 = 0.1$

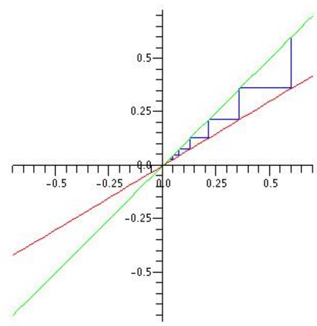


Figure: $m = 0.6; x_0 = 0.6$

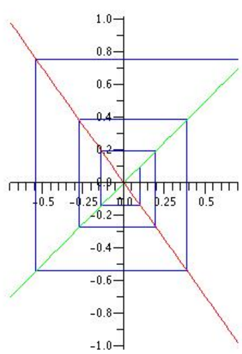


Figure: $m = -1.4; x_0 = 0.1$

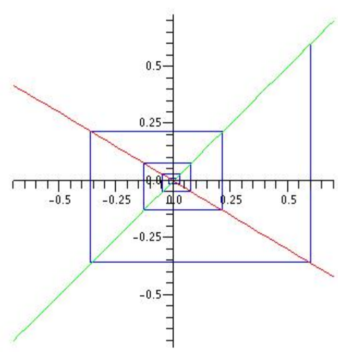


Figure: $m = -0.6; x_0 = 0.6$

If $|L'(0)| > 1$ then $x = 0$ is a repelling fixed point; if $|L'(0)| < 1$ then $x = 0$ is an attracting fixed point. Consequently, we make the following definition . . . .

Definition of Attracting and Repelling Fixed Points

Definition: Suppose p is a fixed point of F. Then

1. p is an attracting fixed point of F if $|F'(p)| < 1$;
2. p is a repelling fixed point of F if $|F'(p)| > 1$;
3. p is a neutral fixed point of F if $|F'(p)| = 1$. A neutral fixed point is sometimes called an indifferent fixed point.

Example 1

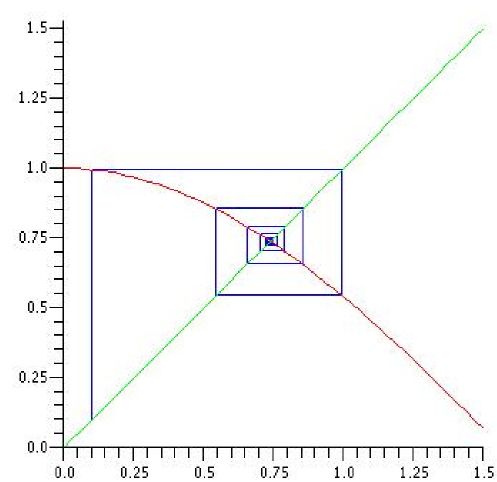
Let $C(x) = \cos x$. C has a single fixed point p in the interval $[0, \pi/2]$. It is an attracting fixed point since

$$|C'(p)| = |-\sin p| < 1.$$

We already know

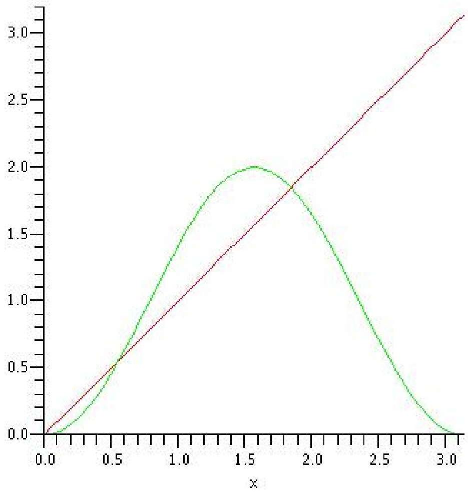
$$p = 0.739085133\ldots$$

At right is the cobweb representation of the orbit of $x_0 = 0.1$ under C.



Example 2

Let $F(x) = 2(\sin x)^2$. Its graph is below; there are 3 fixed points.



$$F'(x) = 4\sin x\cos x = 2\sin 2x.$$

Then

$$\begin{aligned} |F'(x)| < 1 \quad &\Leftrightarrow \quad |2\sin 2x| < 1 \\ &\Leftrightarrow \quad |\sin 2x| < \frac{1}{2} \\ &\Leftrightarrow \quad |x| < \frac{\pi}{12} \end{aligned}$$

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So the only attracting fixed point of F is $x = 0$.

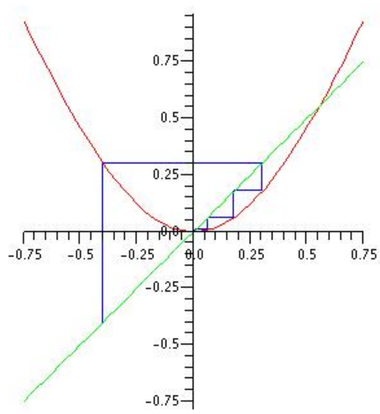

Figure: $x_0 = -0.4$

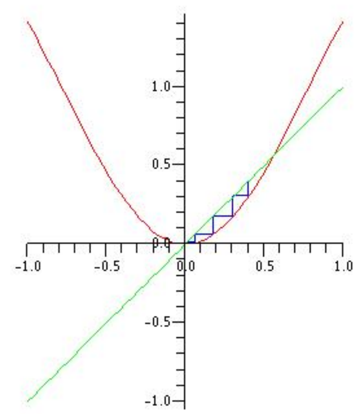

Figure: $x_0 = 0.4$

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The Repelling Fixed Points of Example 2

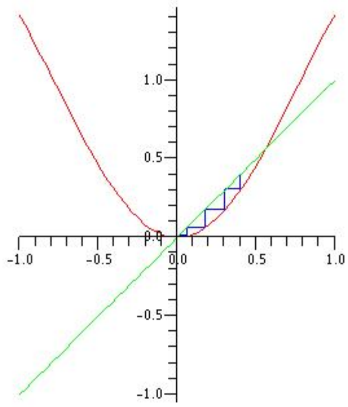


Figure: $x_0 = 0.4$

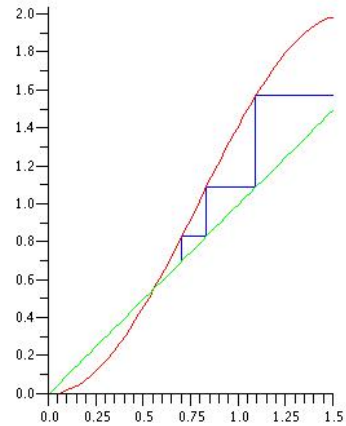


Figure: $x_0 = 0.7$

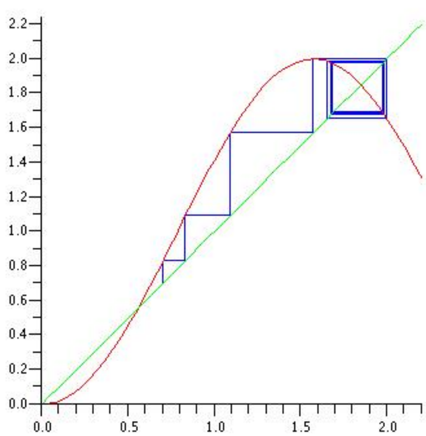


Figure: $x_0 = 0.7$

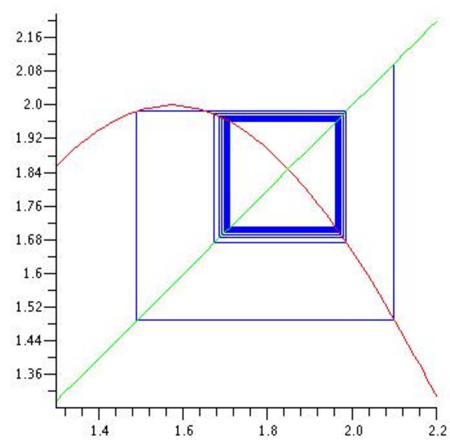
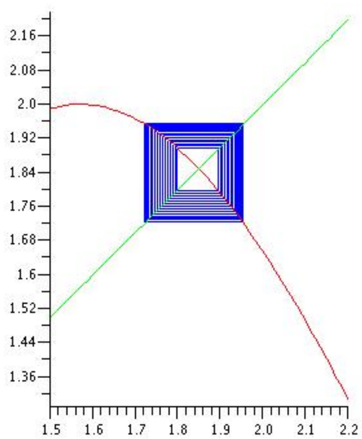
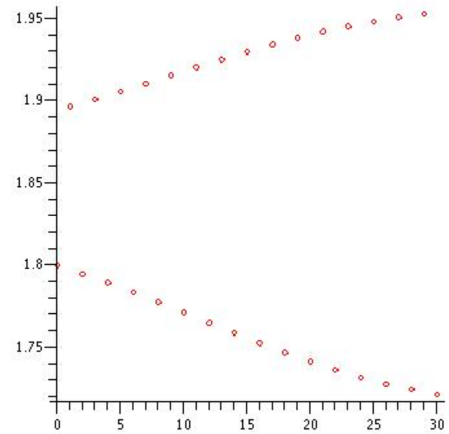


Figure: $x_0 = 2.1$

The third fixed point is also repelling, but the orbits do not tend to infinity. They seem to tend towards a cycle of period 2.

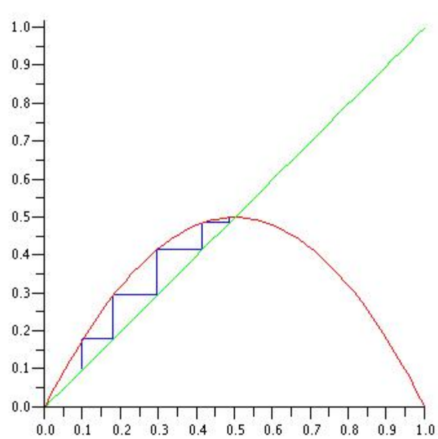
Both of the following graphical representations show that the orbit of $x_0 = 1.8$ under F is not attracted to the fixed point at approximately $p = 1.85$, but is repelled from it to an apparent 2-cycle.

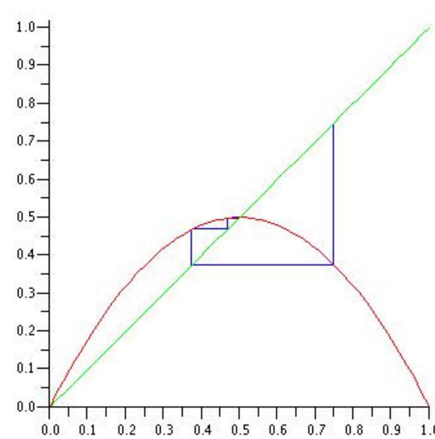




Example 3: $F(x) = 2x - 2x^2$.

$F(x) = x \Leftrightarrow x = 2x^2 \Leftrightarrow x = 0$ or $x = 1/2$, and $F'(x) = 2 - 4x$. So $x = 0$ is a repelling fixed point and $x = 1/2$ is an attracting fixed point.

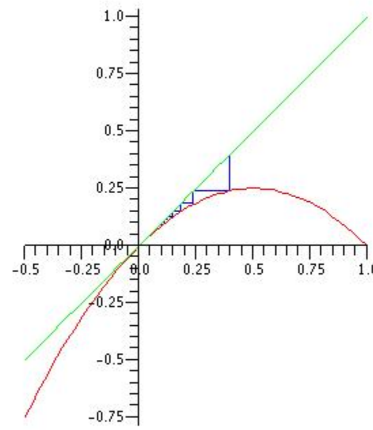




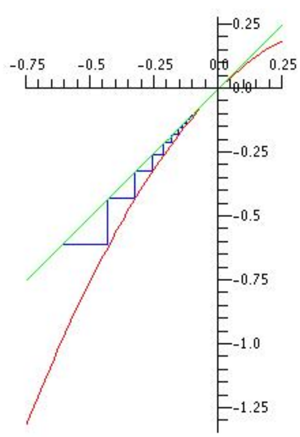
Example 4: $F(x) = x - x^2$.

Then $F(x) = x \Leftrightarrow x^2 = 0 \Leftrightarrow x = 0$; and $F'(x) = 1 - 2x$.

Now $F'(0) = 1$; so the only fixed point of F is a neutral fixed point.
If $x_0 = .4$, the orbit of x_0 under F is attracted to 0.



But if $x_0 = -0.1$, then the orbit of x_0 under F is repelled from 0:



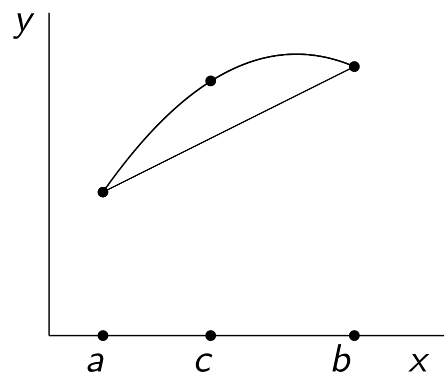
There are many other possibilities for orbits of seeds close to a neutral fixed point! See the examples in the book, and the exercises.

Calculus Review: The Mean Value Theorem

Let f be continuous on $[a, b]$, differentiable on (a, b).

Then there is a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Whether or not a fixed point is attracting or repelling is a consequence of the Mean Value Theorem.

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Attracting Fixed Point Theorem

Suppose p is an attracting fixed point for F. There is an interval I that contains p in its interior such that if x_0 is any point in I then

1. $x_n = F^n(x_0)$ *is in* I*;*
2. $\lim_{n\to\infty} x_n = p$.

That is, for all x_0 sufficiently close to the attracting fixed point p, every orbit of x_0 under F converges to p.

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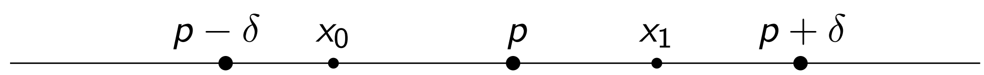
Proof of the Attracting Fixed Point Theroem

Since $|F'(p)| < 1$, there is a number λ such that $0 < \lambda < 1$ and a number $\delta > 0$ such

$$x \in [p-\delta, p+\delta] \Rightarrow |F'(x)| \leq \lambda < 1.$$

Let $I = [p-\delta, p+\delta]$, and let $x_0 \in I$.



By MVT there is a number $x \in I$ such that

$$\begin{aligned} |F(x_0) - F(p)| = |F'(x)||x_0 - p| &\leq \lambda|x_0 - p| \\ \Rightarrow |x_1 - p| &\leq \lambda|x_0 - p| \end{aligned}$$

That is, x_1 is closer to p than x_0 is, since $\lambda < 1$.

Since x_1 is also in I, repeat the above argument to conclude that

$$|x_2 - p| \leq \lambda|x_1 - p| \Rightarrow |x_2 - p| \leq \lambda^2|x_0 - p|.$$

By repeatedly using the same argument, for any $n > 0$: $x_n \in I$ and

$$|x_n - p| \leq \lambda^n|x_0 - p|.$$

This means the orbit of x_0 under F is entirely in the interval I and

$$\lim_{n\to\infty} x_n = p,$$

since $\lambda < 1$.

Repelling Fixed Point Theorem

Suppose p is a repelling fixed point for F. There is an interval I that contains p in its interior such that if $x \in I, x \neq p$, then $|F(x) - p| > |x - p|$.

Proof: $|F'(p)| > 1$, so there are numbers λ and δ such that

$$|F'(p)| > \lambda > 1,$$

and $x \in [p - \delta, p + \delta] \Rightarrow |F'(x)| \geq \lambda > 1$. Let $I = [p - \delta, p + \delta]$, and let $x \in I$. By MVT there is a number $c \in I$ such that

$$|F(x) - F(p)| = |F'(c)||x - p| \geq \lambda|x - p| \ \Rightarrow |F(x) - p| \geq \lambda|x - p|.$$

Since $\lambda > 1$ it follows that $|F(x) - p| > |x - p|$.

Consequences of the Repelling Fixed Point Theorem

1. That is, $F(x)$ is farther from p then x is. So no matter how close $x_n \neq p$ is to p, $x_{n+1} = F(x_n)$ is farther from p.
2. Indeed, as Devaney points out: if x_0 is any point in I, $x_0 \neq p$, then there is an integer $n > 0$ such that

$$x_n = F^n(x_0) \notin I.$$

That is, for all x_0 sufficiently close to but not equal to the repelling fixed point p, the orbit of x_0 under F cannot remain in the interval I. Why? If $x_0, x_1, \ldots, x_n$ are all in I then, by repeating the argument in the proof above n times, $|x_n - p| \geq \lambda^n|x_0 - p|$. For n big enough $\lambda^n|x_0 - p|$ will be greater than the length of the interval I; since $\lambda > 1$. Thus not every point in the orbit of x_0 under F can be in I.

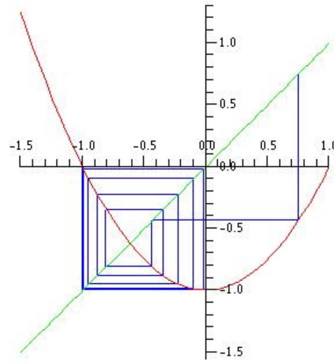
Example 1: An Attracting 2-Cycle

Let $F(x) = x^2 - 1$. It has two fixed points, $x = \dfrac{1 \pm \sqrt{5}}{2}$.

But both fixed points are repelling, since $F'(x) = 2x$ and

$$\left| F'\left(\frac{1 \pm \sqrt{5}}{2}\right) \right| > 1.$$

What happens to the orbit of $x_0 = 0.75$ under F?



It seems to be attracted to the 2-cylce $0, -1$. Why?

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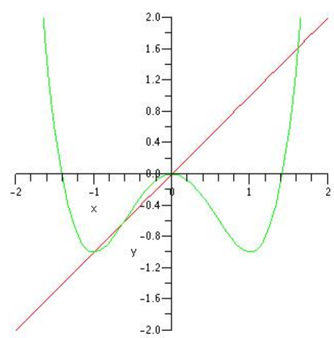
Example 1, Continued

$F^2(x) = (x^2 - 1)^2 - 1 = x^4 - 2x^2$;

$$\begin{aligned} & F^2(x) = x \\ \Leftrightarrow \quad & x^4 - 2x^2 - x = 0 \\ \Leftrightarrow \quad & x(x+1)(x^2 - x - 1) = 0 \\ \Leftrightarrow \quad & x = 0, x = -1, x = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$



$(F^2)'(x) = 4x^3 - 4x \Rightarrow (F^2)'(0) = 0, (F^2)'(-1) = 0$, and $(F^2)'\left(\dfrac{1 \pm \sqrt{5}}{2}\right) = 6 \pm 2\sqrt{5}$. So the only attracting fixed points of F^2 are $x = 0$ and $x = -1$.

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Example 1, Continued

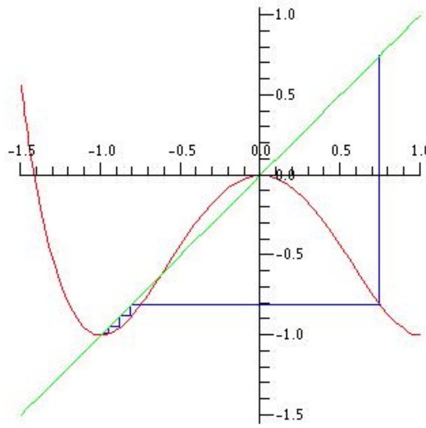

Figure: Orbit of $x_0 = 0.75$ under F^2 converges to -1.

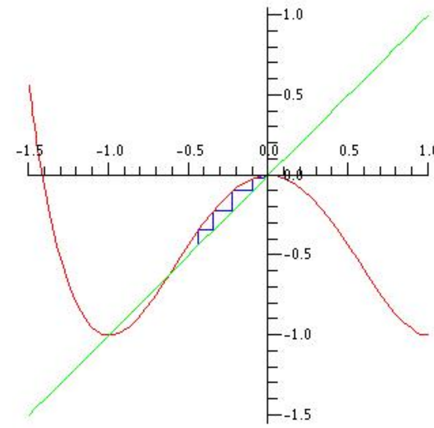

Figure: Orbit of $x_0 = -0.4375$ under F^2 converges to 0.

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Example 1, Concluded

The orbit of $x_0 = .75$ under F^2 converges to -1; but the orbit of $x_1 = F(0.75) = -0.4375$ under F^2 converges to 0. That is, orbits under F^2 are attracted to the fixed points of F^2. However, orbits under F will cycle back and forth as they converge to the 2-cycle consisting of the attracting fixed points of F^2.

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Chain Rule Along a Cycle

A simple chain rule calculation gives the key to determining if an n-cycle of F is attracting or repelling. Suppose $x_0, x_1, \ldots, x_{n-1}$, with $x_i = F^i(x_0)$, lie on a cycle of period n for F. Then

$$\begin{aligned}
(F^n)'(x_0) &= (F \circ F^{n-1})'(x_0) \\
&= F'(F^{n-1}(x_0))(F^{n-1})'(x_0) \\
&= F'(x_{n-1})(F \circ F^{n-2})'(x_0) \\
&= F'(x_{n-1})F'(F^{n-2}(x_0))(F^{n-2})'(x_0) \\
&= F'(x_{n-1})F'(x_{n-2})(F \circ F^{n-3})'(x_0) \\
&\ldots \quad \ldots \\
&= F'(x_{n-1})F'(x_{n-2})\ldots F'(x_2)F'(x_1)F'(x_0)
\end{aligned}$$

Note: $(F^n)'(x_i)$ is the same for any point on the n-cycle.

Types of Cycles

Suppose $x_0, x_1, \ldots, x_{n-1}$, with $x_i = F^i(x_0)$, lie on a cycle of period n for F. Let x_i be any point in the n-cycle, $0 \leq i \leq n-1$.

1. The cycle is attracting if $|(F^n)'(x_i)| < 1$. That is
$$\left|F'(x_{n-1})F'(x_{n-2})\ldots F'(x_2)F'(x_1)F'(x_0)\right| < 1.$$
2. The cycle is repelling if $|(F^n)'(x_i)| > 1$. That is
$$\left|F'(x_{n-1})F'(x_{n-2})\ldots F'(x_2)F'(x_1)F'(x_0)\right| > 1.$$
3. The cycle is neutral if $|(F^n)'(x_i)| = 1$. That is
$$\left|F'(x_{n-1})F'(x_{n-2})\ldots F'(x_2)F'(x_1)F'(x_0)\right| = 1.$$

Example 2

Let $F(x) = x^2 - 1$, as in Example 1. F has a 2-cycle 0 and -1 since

$$F(0) = -1 \text{ and } F(-1) = 0.$$

$F'(x) = 2x$, so

$$F'(0)F'(-1) = (0)(-2) = 0 < 1,$$

and the 2-cycle is attracting.

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Example 3

Let $F(x) = -\frac{3}{2}x^2 + \frac{5}{2}x + 1$. F has a 3-cycle $0, 1$ and 2 since

$$F(0) = 1, F(1) = 2, \text{ and } F(2) = 0.$$

$F'(x) = -3x + \frac{5}{2}$, so

$$F'(0)F'(1)F'(2) = \left(\frac{5}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{7}{2}\right) = \frac{35}{8} > 1,$$

and the 3-cycle is repelling.

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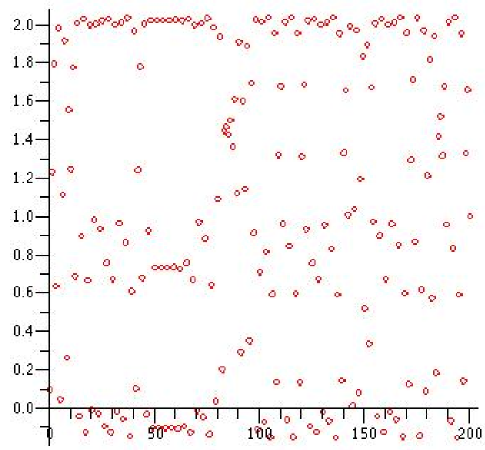
An Orbit For Example 3

Here is the orbit of $x_0 = 0.1$ under F. Even though the seed is close to the periodic point 0, the orbit does not tend toward the 3-cycle, even after 200 iterations:



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Example 4

Even though the doubling function D has many cycles every one of them must be repelling. Why? Recall

$$D : [0,1) \longrightarrow [0,1)$$

by

$$D(x) = \begin{cases} 2x & \text{if} \quad 0 \leq x < \frac{1}{2} \\ 2x-1 & \text{if} \quad \frac{1}{2} \leq x < 1 \end{cases}.$$

So

$$D'(x) = 2, \text{ if } x \neq \frac{1}{2}.$$

If x_0 is any periodic point with prime period n, then

$$(D^n)'(x_0) = 2^n > 1.$$

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The Quadratic Map Q_c

Let

$$Q_c(x) = x^2 + c,$$

where c is a constant. For each different value of c we get a different dynamical system Q_c. In Chapters 6 to 10 we shall do a thorough analysis of the different dynamical systems Q_c as the parameter c varies. And to end the course, in Chapters 16 and 17, we will look at the quadratic map again, but as a function of a complex variable z:

$$Q_c(z) = z^2 + c,$$

where c will be a complex number. But for the moment we limit ourselves to real variables x and real numbers c.

The Fixed Points of Q_c.

$$\begin{aligned} Q_c(x) = x \quad &\Leftrightarrow \quad x^2 + c = x \\ &\Leftrightarrow \quad x^2 - x + c = 0 \\ &\Leftrightarrow \quad x = \frac{1 \pm \sqrt{1-4c}}{2} \end{aligned}$$

Let

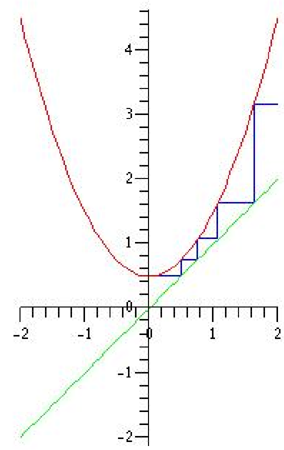
$$p_+ = \frac{1+\sqrt{1-4c}}{2} \text{ and } p_- = \frac{1-\sqrt{1-4c}}{2}.$$

There are three cases:

1. Q_c has no (real) fixed points if $c > 1/4$
2. $p_+ = p_- = 1/2$ if $c = 1/4$
3. p_+ and p_- are real and distinct if $c < 1/4$

The Case $c > 1/4$

In this case the graph of Q_c never intersects the line $y = x$. For any choice of x_0 the orbit of x_0 under Q_c will tend to infinity. The graphical analysis to the right exhibits the case for $c = .5$ and $x_0 = 0$.



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The case $c = 1/4$

In this case there is only one fixed point, $p = 1/2$, and it is a neutral fixed point since $Q_c'(x) = 2x \Rightarrow Q_c'(1/2) = 1$.

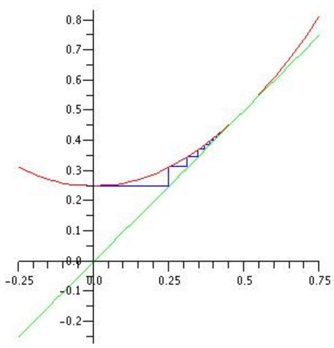

Figure: $x_0 = 0$

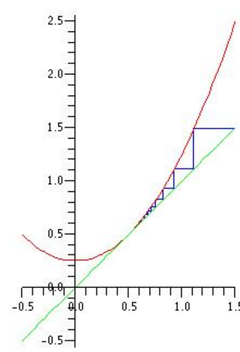

Figure: $x_0 = 0.6$

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The Case $c < 1/4$

Since $Q_c'(x) = 2x$ we have

$$Q_c'(p_+) = 2p_+ = 1 + \sqrt{1-4c} > 1.$$

So p_+ is always a repelling fixed point for Q_c. On the other hand,

$$Q_c'(p_-) = 2p_- = 1 - \sqrt{1-4c}.$$

Thus p_- will be attracting for some values of c and repelling for others. Some terminology: the quadratic family Q_c has a tangent, or saddle-node, bifurcation at $c = 1/4$. That is, for $c > 1/4$ there are no fixed points for Q_c; at $c = 1/4$ there is a neutral fixed point for Q_c; and if $c < 1/4$ there are two fixed points for Q_c, one attracting and one repelling.

The Fixed Point p_- with $c < 1/4$

1. p_- is an attracting fixed point for Q_c if

$$\begin{aligned} |Q_c'(p_-)| < 1 &\Leftrightarrow -1 < 1 - \sqrt{1-4c} < 1 \\ &\Leftrightarrow -2 < -\sqrt{1-4c} < 0 \\ &\Leftrightarrow 2 > \sqrt{1-4c} > 0 \\ &\Rightarrow 4 > 1-4c > 0 \\ &\Leftrightarrow 3 > -4c > -1 \\ &\Leftrightarrow -3/4 < c < 1/4 \end{aligned}$$

2. p_- is a neutral fixed point for Q_c if $c = -3/4$
3. p_- is a repelling fixed point for C_c if $c < -3/4$.

The Interval $[-p_+, p_+]$, for $c \leq 1/4$

Since $Q_c(-x) = Q_c(x)$, the fate of any orbit of $-x_0$ under Q_c is always the same as the fate of the orbit of x_0 under Q_c. In particular,

$$Q_c(-p_+) = Q_c(p_+) = p_+,$$

so $-p_+$ is an eventual fixed point for Q_c. We have seen that for $x_0 > p_+$ the orbit of x_0 under Q_c tends to infinity. Likewise, if $x_0 < -p_+$ the orbit of x_0 under Q_c will also tend to infinity. Thus for any seed

$$x_0 \notin [-p_+, p_+]$$

the orbit under Q_c tends to infinity. The next slide illustrates this result for the case $c = -1/4$ and $|x_0| = 1.25$. Aside: if $c = -1/4$,

$$p_+ = \frac{1+\sqrt{2}}{2} \simeq 1.21.$$

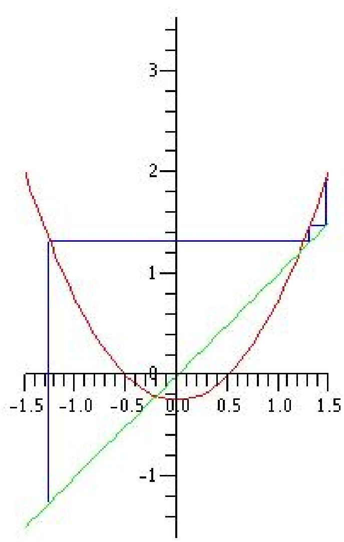

Figure: $c = -1/4; x_0 = -1.25$

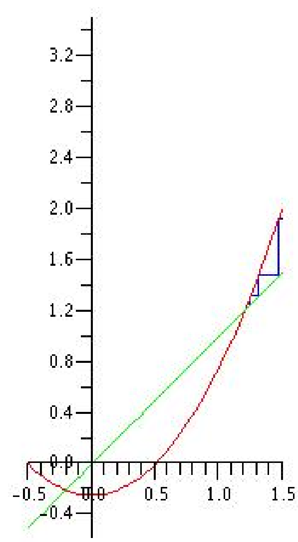

Figure: $c = -1/4; x_0 = 1.25$

If $x_0 \in [-p_+, p_+]$, for $c \leq 1/4$.

All of the interesting dynamics occurs if $x_0 \in [-p_+, p_+]$, for $c \leq 1/4$. Note that

$$p_- = \frac{1 - \sqrt{1-4c}}{2} \in [-p_+, p_+],$$

for $c \leq 1/4$. But, as we have seen above, p_- is only an attracting fixed point if

$$-\frac{3}{4} < c < \frac{1}{4}.$$

It can be proved that if $-3/4 \leq c < 1/4$ and $x_0 \in [-p_+, p_+]$, then the orbit of x_0 under Q_c converges to the fixed point p_-. Two examples are illustrated in the next slide.

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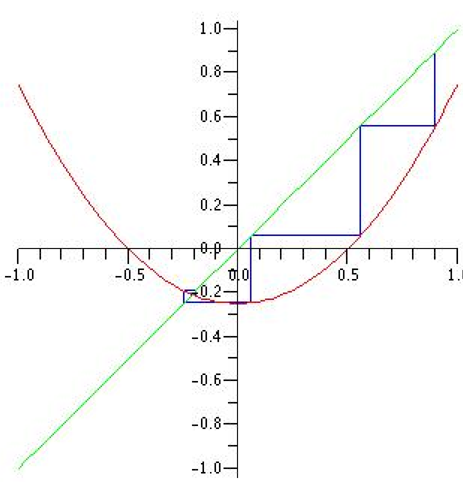

Figure: $c = -1/4; x_0 = 0.9$

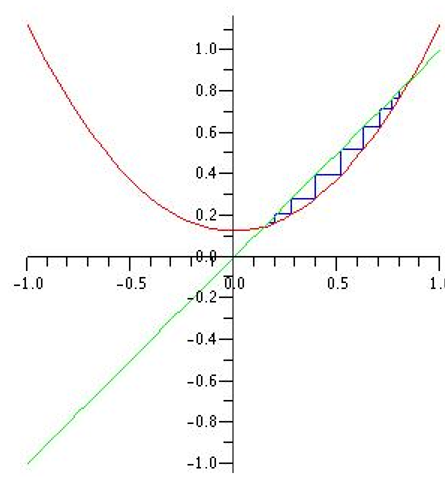

Figure: $c = 1/8; x_0 = 0.8$

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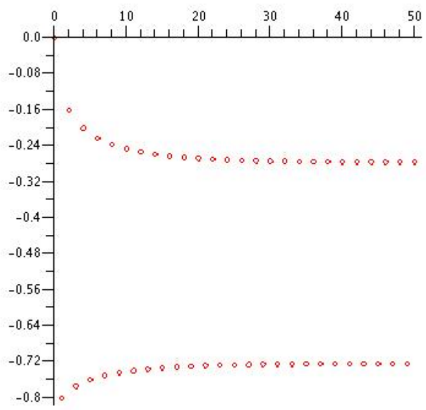
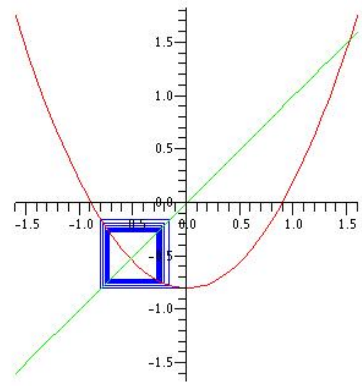
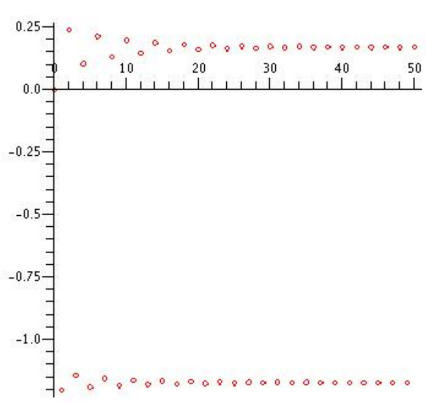
An Attracting 2-Cycle

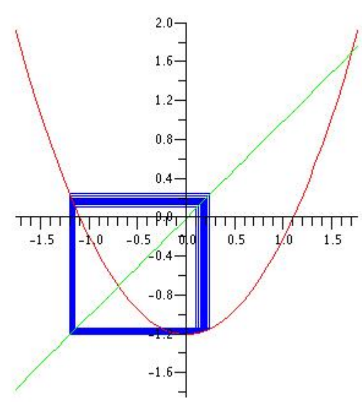
What happens if $c \leq -3/4$? Here are two graphical representations of the orbit of $x_0 = 0$ under $Q_{-4/5}$:





Here are two more graphical representations, of the orbit of $x_0 = 0$ under $Q_{-6/5}$:





For both values, $c = -4/5$ or $c = -6/5$, the orbit appears to converge to an attracting 2-cycle. Why should this be?

Points of Prime Period 2, $c \leq -3/4$

$$
\begin{aligned}
Q_c^2(x) = x \quad &\Leftrightarrow \quad (x^2+c)^2 + c = x \\
&\Leftrightarrow \quad x^4 + 2cx^2 - x + c^2 + c = 0 \\
&\Leftrightarrow \quad (x^2 - x + c)(x^2 + x + c + 1) = 0 \text{ (Why?)} \\
&\Leftrightarrow \quad x = p_-, x = p_+ \text{ or } x = \frac{-1 \pm \sqrt{-3-4c}}{2}
\end{aligned}
$$

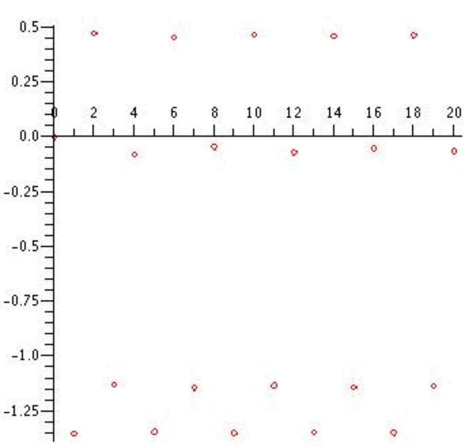
Let

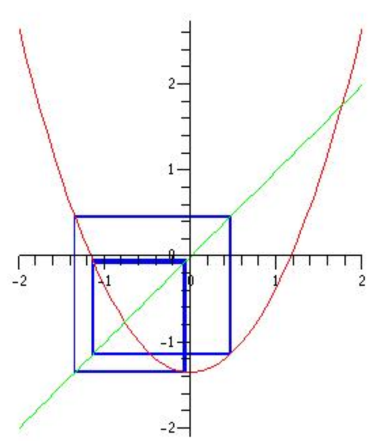
$$
q_- = \frac{-1 - \sqrt{-3-4c}}{2}, \quad q_+ = \frac{-1 + \sqrt{-3-4c}}{2}.
$$

Check that $Q_c'(q_-)Q_c'(q_+) = 4 + 4c$. Hence q_-, q_+ is an attracting two cycle if

$$
|4 + 4c| < 1 \Leftrightarrow -\frac{5}{4} < c < -\frac{3}{4}.
$$

The quadratic family Q_c is said to have a period-doubling bifurcation at $c = -3/4$. If $c < -5/4$, then the 2-cycle q_-, q_+ becomes a repelling 2-cycle. The dynamics then change again: there is another period-doubling bifurcation at $c = -5/4$. Here is an example, with $x_0 = 0$ and $c = -1.35$:





Definition of a Saddle-Node Bifurcation

A one-parameter family of functions F_λ has a tangent, or saddle-node, bifurcation in the open interval $I \subset \mathbb{R}$ at the parameter value λ_0 if there is an $\epsilon > 0$ such that

1. F_{λ_0} has one fixed point in I and this fixed point is neutral;
2. for all λ in one half of the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, F_λ has no fixed points in I;
3. for all λ in the other half of the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, F_λ has two fixed points in I, one attracting and one repelling.

Note: this definition describes a change in the fixed point structure of F_λ. Periodic points of F_λ can also have a tangent bifurcation: replace F_λ with F_λ^n for a cycle of period n in the above definition.

Example 1

The quadratic family Q_c has a tangent bifurcation at $c_0 = 1/4$, since

1. $Q_{1/4}(x) = x^2 + 1/4$ has a fixed point, $p = 1/2$, which is a neutral fixed point;
2. Q_c has no fixed points for $1/4 < c < \infty$;
3. Q_c has two fixed points p_- and p_+ if $-3/4 < c < 1/4$, of which p_- is attracting and p_+ is repelling.

In terms of the definition, you can take $I = \mathbb{R}$ and $\epsilon = 1$.

Graphs for Example 1

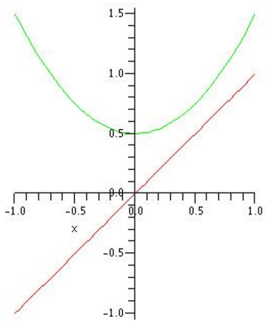


Figure: $c = 1/2$

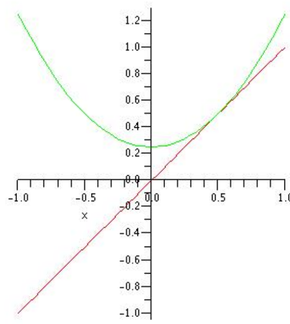


Figure: $c = 1/4$

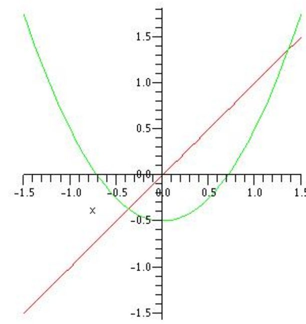


Figure: $c = -1/2$

From this sequence of graphs you can see why its called a tangent bifurcation.

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Bifurcation Diagram for Example 1

For $c \leq 1/4$ we had

$$p_+ = \frac{1 + \sqrt{1-4c}}{2}$$

and

$$p_- = \frac{1 - \sqrt{1-4c}}{2}.$$

The diagram to the right plots p_+, p_- as functions of c.

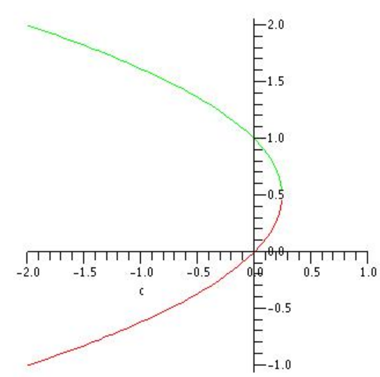

Figure: Bifurcation at $c_0 = 1/4$

This is called a bifurcation diagram.

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Example 2

Let $E_\lambda(x) = e^x + \lambda$; this is called the exponential family. It has a tangent bifurcation in the interval $\mathbb{R}$ at $\lambda = -1$:

1. If $\lambda > -1$ then $E_\lambda(x) > x$ for all x, so E_λ has no fixed points.
2. If $\lambda = -1$ then the equation of the tangent line to E_{-1} at $x = 0$ is $y = x$. Since $E'_{-1}(0) = e^0 = 1$, the fixed point $x = 0$ is a neutral fixed point.
3. If $\lambda < -1$ then the graph of E_λ intersects the line with equation $y = x$ in two points, say p_1 and p_2, with $p_1 < p_2$. Check that: $p_1 < 0$ and $p_2 > 1$;

$$0 < E'_\lambda(p_1) = e^{p_1} < 1; E'_\lambda(p_2) = e^{p_2} > 1;$$

so p_1 is always an attracting fixed point, and p_2 is always a repelling fixed point. In this example, any $\epsilon > 0$ is OK.

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Graphs for Example 2

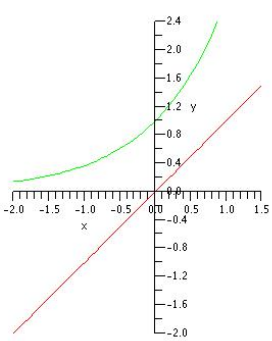

Figure: $\lambda = 0$

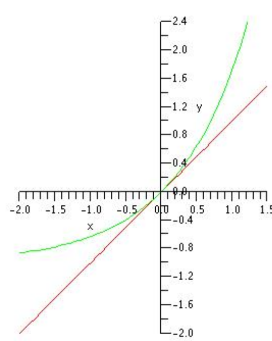

Figure: $\lambda = -1$

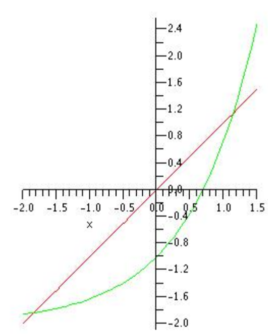

Figure: $\lambda = -2$

It is not easy to draw the bifurcation diagram for the exponential family, because you can't solve for x if $e^x + \lambda = x$ and $\lambda < -1$.

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Example 3

$F_\lambda(x) = \lambda x(1-x), \lambda \neq 0$ is called the logistic family. Its fixed points are easy to determine:

$$\begin{aligned} F_\lambda(x) = x \quad &\Leftrightarrow \quad \lambda x(1-x) = x \\ &\Leftrightarrow \quad x = 0 \text{ or } \lambda - \lambda x = 1 \\ &\Leftrightarrow \quad x = 0 \text{ or } x = \frac{\lambda - 1}{\lambda} = 1 - \frac{1}{\lambda} \end{aligned}$$

Now

$$F_\lambda'(0) = \lambda \text{ and } F_\lambda'\left(1 - \frac{1}{\lambda}\right) = 2 - \lambda,$$

as you may check. It can also be shown that for $0 < \lambda \leq 4$,

$$F_\lambda : [0,1] \longrightarrow [0,1].$$

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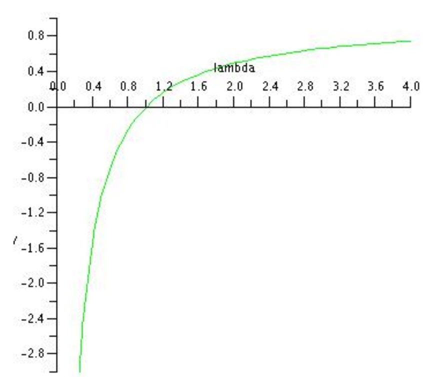
If $\lambda = 1$ then the only fixed point of $F_\lambda(x)$ is $x = 0$ and it is neutral. If $0 < \lambda < 1$ then $x = 0$ is an attracting fixed point and $x = 1 - \dfrac{1}{\lambda}$ is repelling. But if $1 < \lambda < 3$, then $x = 0$ is the repelling fixed point and $x = 1 - \dfrac{1}{\lambda}$ is the attracting fixed point.



According to our definition, there is no tangent bifurcation point for the logistic family at $\lambda = 1$: there are two fixed points on each side of $\lambda = 1$. It makes no difference what you choose for the open interval I or for $\epsilon > 0$; you can't arrange F_λ for λ in one side of $(1-\epsilon, 1+\epsilon)$ to have no fixed points. However it is true that $F_1(x)$ is tangent to the line $y = x$ at $x = 0$.

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Definition of a Period Doubling Bifurcation

A one-parameter family of functions F_λ has a period-doubling bifurcation in the open interval $I \subset \mathbb{R}$ at the parameter value λ_0 if there is an $\epsilon > 0$ such that

1. for each $\lambda \in [\lambda_0 - \epsilon, \lambda_0 + \epsilon]$, F_λ has a unique fixed point $p_\lambda \in I$;
2. for all λ in one half of the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, including $\lambda = \lambda_0$, F_λ has no cycles of period 2 in I and p_λ is attracting (resp. repelling);
3. for all λ in the other half of the interval $(\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, excluding $\lambda = \lambda_0$, there is a unique 2-cycle $q^1_\lambda, q^2_\lambda \in I$ with $F(q^1_\lambda) = q^2_\lambda$. This 2-cycle is attracting (resp. repelling). Meanwhile, the fixed point p_λ is repelling (resp. attracting).
4. As $\lambda \to \lambda_0$ both $q^i_\lambda \to p_\lambda$.

Remarks About the Definition

1. At a period doubling bifurcation one of two things happens: an attracting fixed point changes to a repelling fixed point while at the same time giving rise to an attracting 2-cycle; or a repelling fixed point changes to an attracting fixed point while at the same time giving rise to a repelling 2-cycle.
2. Whereas a tangent bifurcation occurs when the graph of F_λ is tangent to the line with equation $y = x$, a period-doubling bifurcation occurs when the graph of F_λ is perpendicular to the line with equation $y = x$, as will be illustrated by some examples. This implies that the graph of F^2_λ is tangent to the line $y = x$ when the bifurcation occurs:
$$(F^2_{\lambda_0})'(p_{\lambda_0}) = F'_{\lambda_0}(F_{\lambda_0}(p_{\lambda_0}))F'_{\lambda_0}(p_{\lambda_0}) = \left(F'_{\lambda_0}(p_{\lambda_0})\right)^2 = (-1)^2 = 1.$$

Example 1

The quadratic family Q_c has a period doubling bifurcation at $c_0 = -3/4$. Why? Recall that

1. $$p_- = \frac{1 - \sqrt{1-4c}}{2}$$
   is an attracting fixed point for $-3/4 \leq c < 1/4$; and that
2. $$q_- = \frac{-1 - \sqrt{-3-4c}}{2}, \; q_+ = \frac{-1 + \sqrt{-3-4c}}{2}$$
   is an attracting 2-cylce if $-5/4 < c < -3/4$, for which p_- is now repelling.

The following graphical analyses illustrate how the orbit of $x_0 = 0.8$ converges to a fixed point for $Q_{-0.6}$ but is attracted to a 2-cyle for $Q_{-0.8}$:

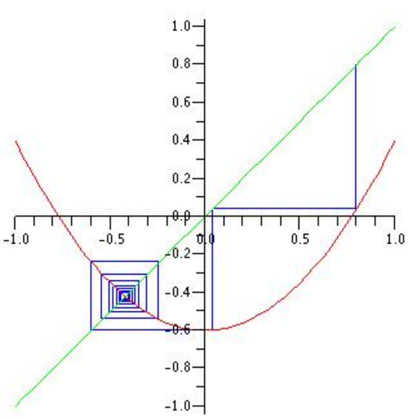

Figure: $c = -0.6; x_0 = 0.8$

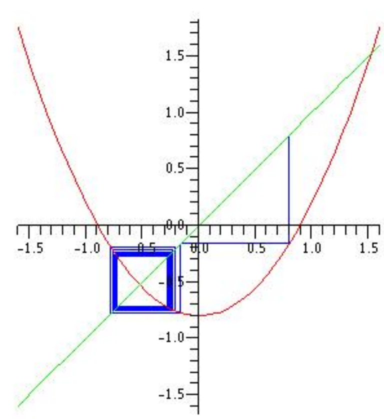

Figure: $c = -0.8; x_0 = 0.8$

The Graphs of Q_c^2 for Example 1

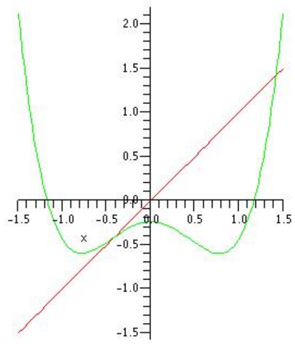

Figure: $c = -3/5$

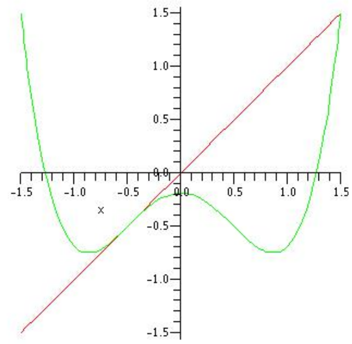

Figure: $c = -3/4$

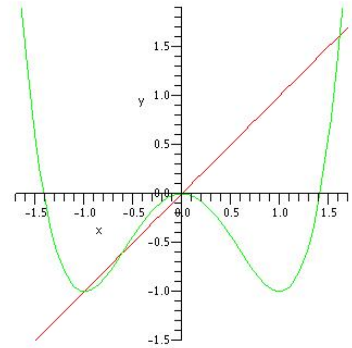

Figure: $c = -1$

$Q^2_{-3/4}(x)$ is tangent to the line $y = x$ at $p_- = -1/2$.

Other Graphs for Example 1

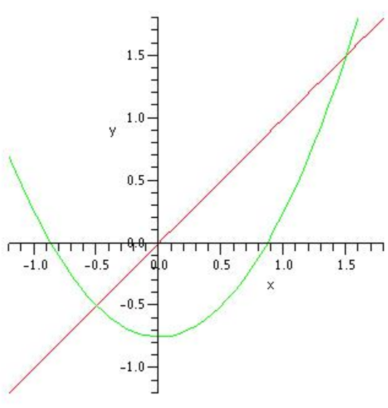

Figure: $Q_{-3/4}$ is perpendicular to the line with equation $y = x$ at $x = -1/2$.

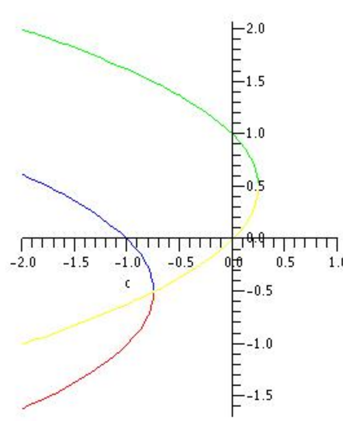

Figure: Bifurcation diagram for Q_c, $-2 < c < 1$

Example 2: the Exponential Family $E_\lambda(x) = e^x + \lambda$

The exponential family has no period doubling bifurcation because it has no points of prime period 2 at all.

$$\begin{aligned} E_\lambda^2(x) = x \quad &\Rightarrow \quad e^{e^x+\lambda} + \lambda = x \\ &\Rightarrow \quad e^{e^x+\lambda} = x - \lambda \\ &\Rightarrow \quad e^x + \lambda = \ln(x - \lambda) \end{aligned}$$

Observe that $E_\lambda^{-1}(x) = \ln(x - \lambda)$. Now, the only intersection points of an increasing function and its inverse function are on the line with equation $y = x$. So every point of period 2 for E_λ is actually a fixed point for E_λ. In general, you can prove that if F is increasing and $F^2(x) = x$ then $F(x) = x$. Suppose $F(x) < x$, then $F^2(x) < F(x) < x$, a contradiction. Similarly if $F(x) > x$.

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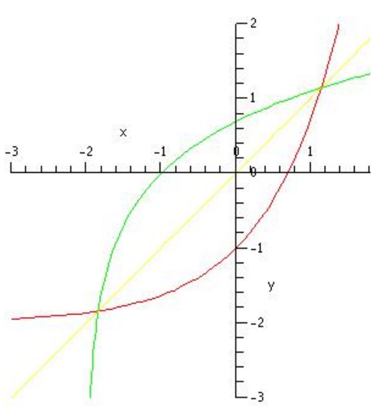


Figure: $E_{-2}(x) = e^x - 2$

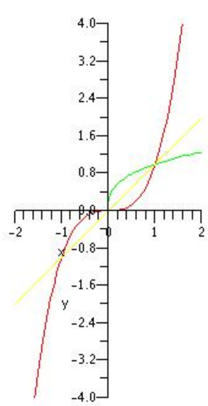


Figure: $F(x) = x^3$

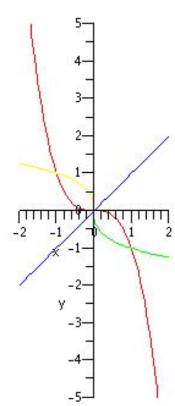


Figure: $G(x) = -x^3$

Neither E_{-2} nor F have points of prime period 2. G is invertible and decreasing and does have a 2-cycle: 1 and -1.

Chapter 5 and 6 Lecture Notes MAT335H1F Lec0101 Burbulla

Example 3: the Logistic Family $F_\lambda(x) = \lambda x(1 - x)$.

There is a period doubling bifurcation at $\lambda_0 = 3$. Check that

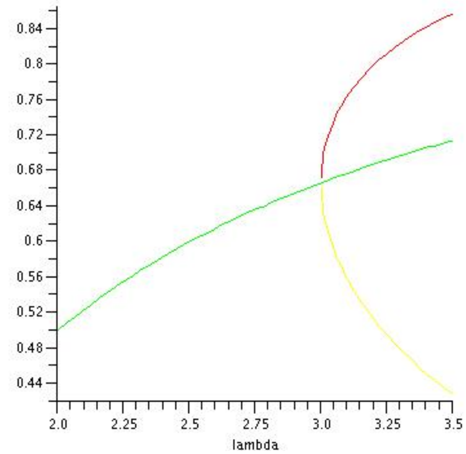
$$p_\lambda = 1 - 1/\lambda;$$

and

$$q_\lambda^1 = \frac{\lambda + 1 + \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda},$$

$$q_\lambda^2 = \frac{\lambda + 1 - \sqrt{\lambda^2 - 2\lambda - 3}}{2\lambda}$$

is the 2-cycle. What should I and ϵ be? Also: see exercises.



Example 4: $F_\lambda(x) = \lambda x - x^3; F'(x) = \lambda - 3x^2$

$$F_\lambda(x) = x \Leftrightarrow x^3 = x(\lambda - 1) \Leftrightarrow x = 0 \text{ or } x = \pm\sqrt{\lambda - 1}.$$

Since

$$F_\lambda'(0) = \lambda \text{ and } F_\lambda'(\pm\sqrt{\lambda - 1}) = 3 - 2\lambda,$$

$x = 0$ is attracting if $-1 < \lambda < 1$ and repelling if $\lambda > 1$ or $\lambda < -1$. The two fixed points

$$x = \pm\sqrt{\lambda - 1}$$

only exist if $\lambda > 1$; they are attracting if $1 < \lambda < 2$ and repelling if $\lambda > 2$.

Devaney's 2-Cycle for $F_\lambda(x) = \lambda x - x^3$

Period 2 points occur if $\lambda > -1$:

$$F_\lambda(x) = -x \Rightarrow F_\lambda^2(x) = F_\lambda(F_\lambda(x)) = F_\lambda(-x) = -F_\lambda(x) = -(-x) = x;$$

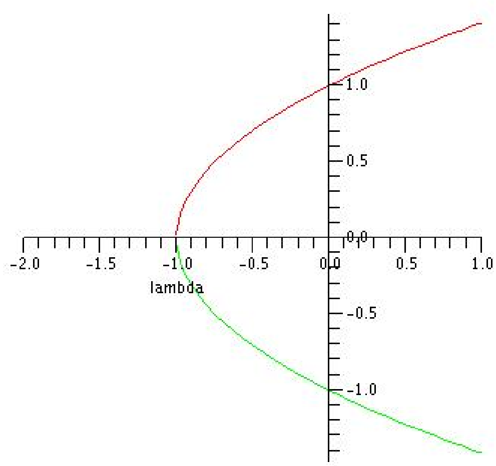
hence the two non-zero solutions to $F_\lambda(x) = -x$, namely

$$x = \pm\sqrt{\lambda+1},$$

form a 2-cycle for F_λ, which is always repelling. See the exercises. So the family F_λ has a period-doubling bifurcation at $\lambda_0 = -1$: the fixed point $x = 0$ is repelling if $\lambda < -1$, attracting if $\lambda > -1$; and there is no 2-cycle for $\lambda < -1$, but there is a repelling 2-cycle if $\lambda > -1$.

Correct Version of Figure 6.15, page 65

Here is the bifurcation diagram for $F_\lambda(x) = \lambda x - x^3$, $-2 \leq \lambda \leq 1$:



For some reason, in the book Figure 6.15 has the functions

$$x = \pm\sqrt{\lambda+1}$$

opening to the left!

The Other 2-Cycles for F_λ

But there are more period 2 points. To find all of them you have to solve $F_\lambda^2(x) = x$:

$$\begin{aligned} & F_\lambda^2(x) = x \\ \Leftrightarrow\quad & \lambda^2 x - \lambda x^3 - \lambda^3 x^3 + 3\lambda^2 x^5 - 3\lambda x^7 + x^9 - x = 0 \\ \Leftrightarrow\quad & x(x^4 - x^2\lambda + 1)(x^2 - \lambda - 1)(x^2 - \lambda + 1) = 0 \\ \Leftrightarrow\quad & x = 0 \text{ or } x^2 = \lambda + 1 \text{ or } x^2 = \lambda - 1 \text{ or } x^2 = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2} \end{aligned}$$

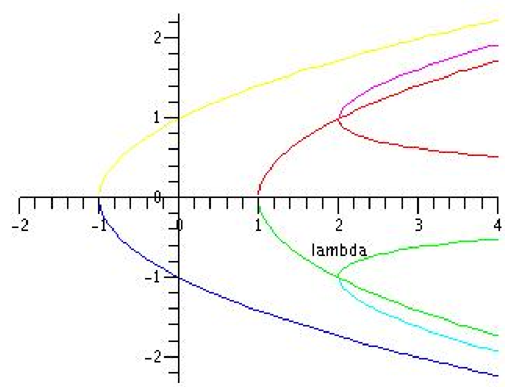
Five of these solutions are the previously calculated fixed points and 2-cycle for F_λ. The other four provide two more 2-cycles:

$$\sqrt{\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}} \text{ and } -\sqrt{\frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}}.$$

Bifurcation Diagram for Example 4, $-2 \le \lambda \le 4$.

Check that:

1. The new 2-cycles only exist if $\lambda \ge 2$.
2. The fixed points $\pm\sqrt{\lambda - 1}$ are attracting if $1 < \lambda < 2$, and repelling if $\lambda > 2$.
3. The new 2-cycles are both attracting if $2 < \lambda < \sqrt{5}$.



So F_λ has two period doubling bifurcations at $\lambda_0 = 2$; one in the interval $I_1 = (0.8, 1.2)$ and one in the interval $I_2 = (-1.2, -0.8)$.