

MAT301H1S Lec5101 Burbulla

Week 12 Lecture Notes

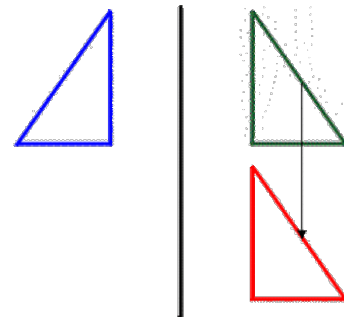
Winter 2020

The Four Types of Isometries of $\mathbb{R}^2$

Recall that the four types of isometries of the plane are:

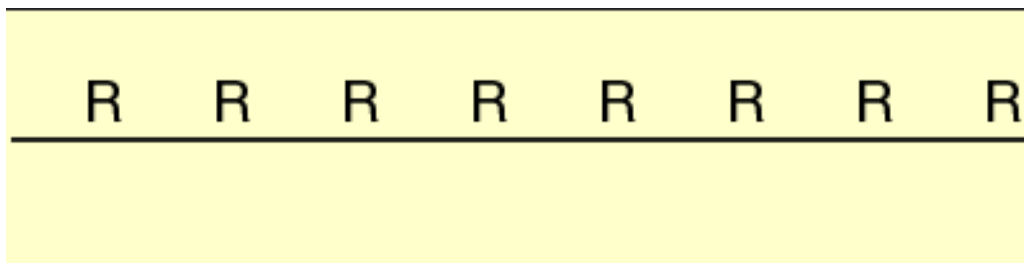
1. **rotations:** $T(\vec{x}) = R_\theta(\vec{x})$ with $[R_\theta] = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$.
2. **reflections:** $T(\vec{x}) = F_\theta(\vec{x})$ with $[F_\theta] = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$, if the reflection is in the line $y = \tan(\theta/2)x$.
3. **translations:** $T(\vec{x}) = \vec{x} + \vec{a}$, for some vector $\vec{a} \in \mathbb{R}^2$.

- glide reflections:** a translation parallel to a line, followed by a reflection in that line; or equivalently, a reflection in a line followed by a translation parallel to the line. The line is called the glide-axis.
- 4.

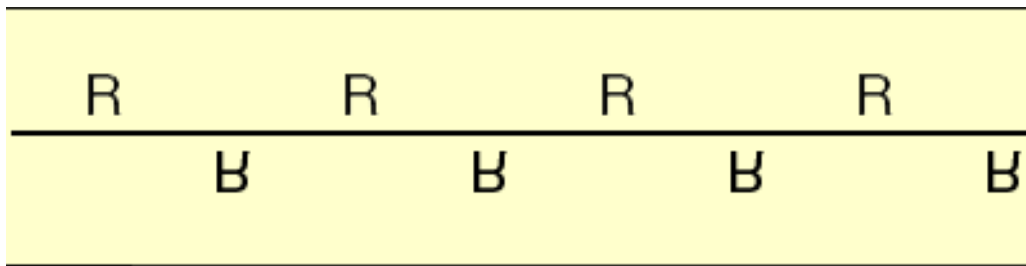


The Frieze Groups

Definition: the **discrete frieze groups** are the symmetry groups of figures in the plane whose subgroup of translations is isomorphic to $\mathbb{Z}$. That is, the figure extends infinitely far in both directions, to the left and the right. The minimum horizontal distance between the repeating element of the pattern defines the generating translation. The simplest case is a figure that has translational symmetry only.



This pattern is called F_1 . Let the translation be T . Its symmetry group is $\langle T \rangle \approx \mathbb{Z}$.

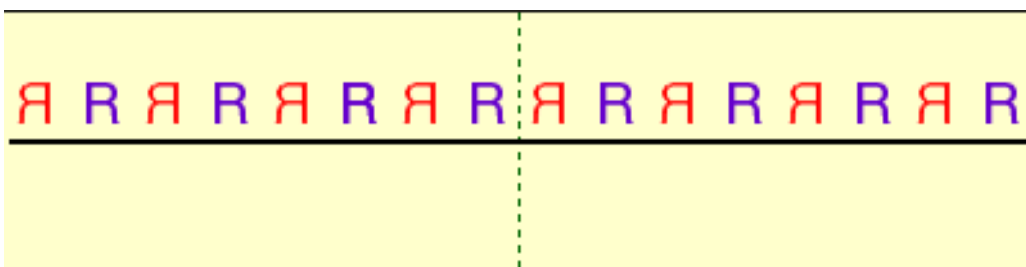
F_2 

Pattern F_2 has two symmetries:

1. a glide reflection G ,
2. and a translation $T = G^2$.

Its symmetry group is also $\mathbb{Z} \approx \langle G \rangle$;

its translation subgroup is $2\mathbb{Z} \approx \langle T \rangle = \langle G^2 \rangle$.

 F_3 

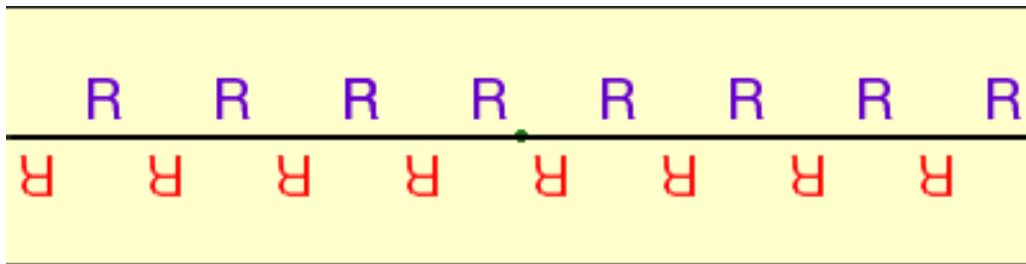
Pattern F_3 also has two symmetries:

1. a vertical reflection V ,
2. and a translation T .

Its symmetry group is

$$\langle TV, V \mid (TV)^2 = V^2 = I, (TV)V = T, |T| = \infty \rangle \approx D_\infty.$$

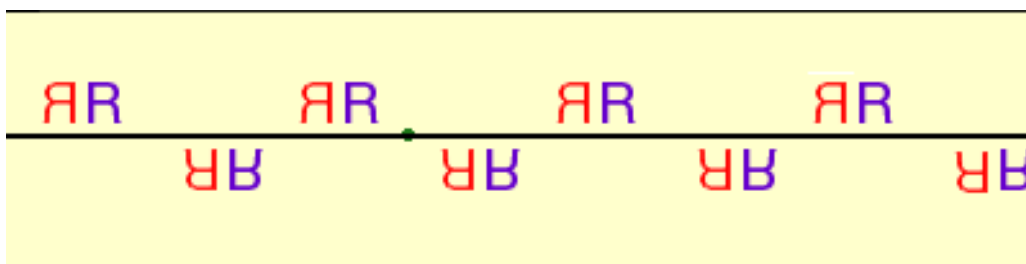
Observe that the distance between consecutive reflection axes is half the length of the shortest translation symmetry.

F_4 

Pattern F_4 also has two symmetries:

1. a rotation of 180° about a point, H , called a half-turn,
2. and a translation T .

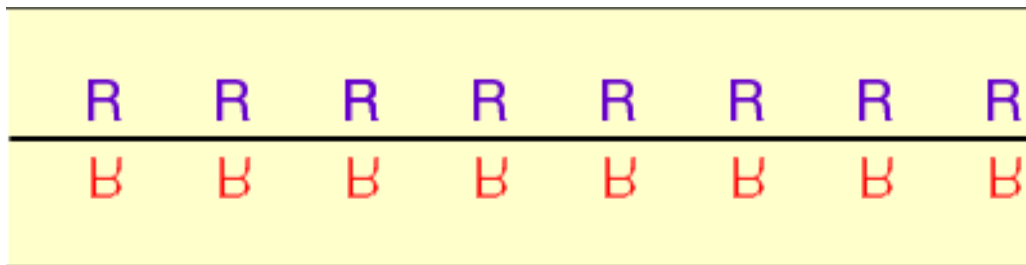
This pattern has the same symmetry group as F_3 , namely D_∞ .

 F_5 

Pattern F_5 has four symmetries:

1. a rotation of 180° about a point, H , called a half-turn,
2. a glide reflection G ,
3. a vertical reflection V ,
4. and a translation T .

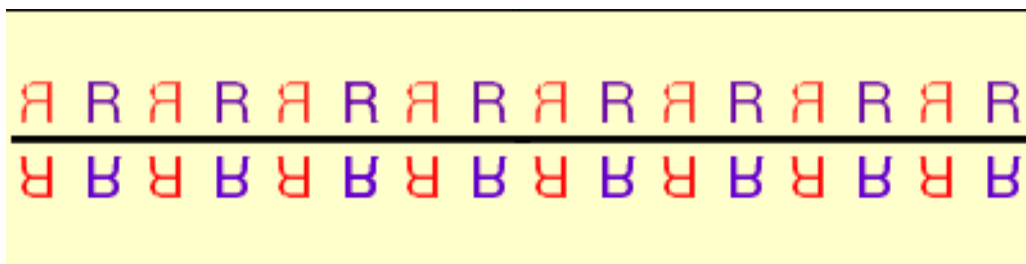
Note that $T = G^2$ and $V = GH$. The symmetry group of this pattern is the same as that of F_3 and F_4 , namely D_∞ .

F_6 

Pattern F_6 has three symmetries:

1. a glide reflection G ,
2. a horizontal reflection F ,
3. and a translation T .

Note that $G = TF$ and that the axis of the glide reflection G is the same as the axis of the horizontal reflection F ; such a glide reflection is called a *trivial* glide reflection. Moreover, $TF = FT$, so the symmetry group of this pattern is Abelian; it is $\mathbb{Z} \oplus \mathbb{Z}_2$.

 F_7 

This pattern exhibits all possible symmetries:

1. a horizontal reflection F ,
2. a glide reflection G ,
3. a half-turn H ,
4. a vertical reflection V ,
5. and a translation T .

$H = FV$ and $G = TF$; the symmetry group of F_7 is $D_\infty \oplus \mathbb{Z}_2$.

Conway's Description of the Seven Frieze Patterns

[Link to Seven Frieze Patterns](#)

The Wallpaper Groups

Now consider patterns in the plane that fill space. In this case the symmetry group of the pattern includes translations in two independent directions; the subgroup of translations that fix the pattern will now be $\mathbb{Z} \oplus \mathbb{Z}$. Such repetitive patterns are called

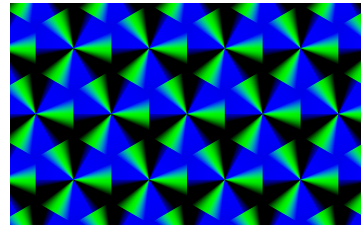
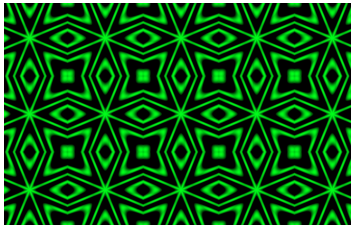
1. wallpaper patterns,
2. tiling patterns,
3. or crystallographic patterns.

The symmetry groups of these patterns are called wallpaper groups or crystallographic groups. There are only 17 possible different wallpaper groups, although we shall not prove this here. All 17 possibilities were known to artisans of antiquity.

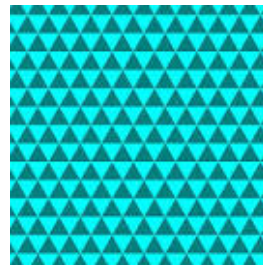
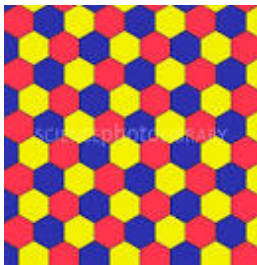
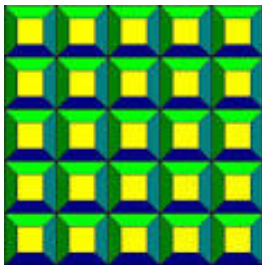
A Quote from Hermann Weyl's *Symmetry*, 1952

One can hardly overestimate the depth of geometric imagination and inventiveness reflected in these patterns. Their construction is far from being mathematically trivial. The art of ornament contains in implicit form the oldest piece of higher mathematics known to us.

Here are three examples:



Some Simple Examples



Notice that the basic shapes in these patterns are the square, the hexagon and the triangle, respectively. This is not a coincidence. Suppose k regular n -gons meet at a vertex in one of these patterns. Then the interior angle of each n -gon is $\pi(1 - 2/n)$ and

$$k\pi \left(1 - \frac{2}{n}\right) = 2\pi \Leftrightarrow \frac{1}{2} = \frac{1}{n} + \frac{1}{k}.$$

The only solutions in positive integers are $(k, n) = (4, 4), (3, 6)$ or $(6, 3)$. No wallpaper pattern can ever be based on a pentagon!

The Seventeen Wallpaper Patterns

The simplest case is when the wallpaper pattern includes a six-fold rotation. There are two possibilities:

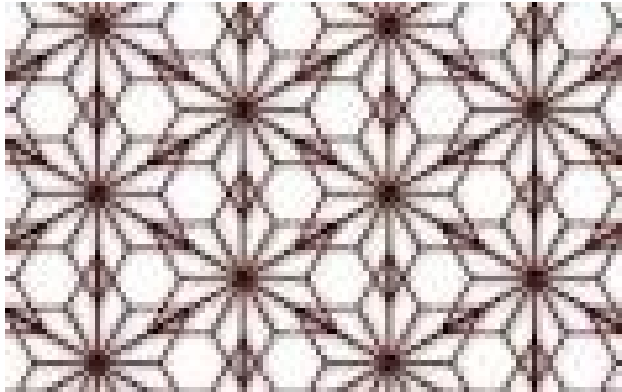


Table: p6m

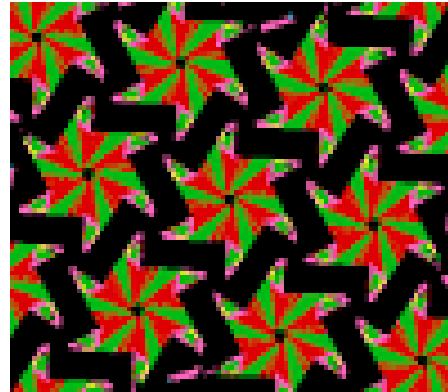


Table: p6

One has axes of reflection; the other one doesn't.

Wallpaper Patterns with 4-Fold Rotational Symmetry

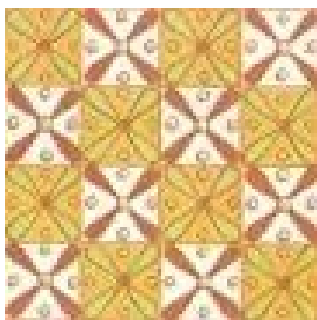


Table: p4m



Table: p4g



Table: p4

How do these patterns differ? The one labeled p4 has only rotational symmetry. The one labeled p4m has rotational symmetry and four reflections, one horizontal, one vertical, and two diagonal. The one labeled p4g has vertical and horizontal reflections, but instead of diagonal reflections has glide-reflections.

Wallpaper Patterns with 3-Fold Rotational Symmetry

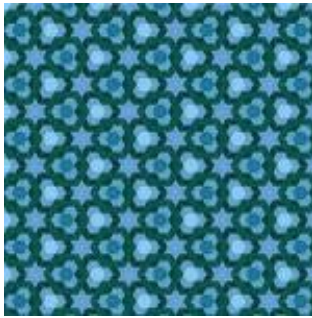


Table: p3m1



Table: p31m



Table: p3

The one labeled p3 has only rotational symmetry. The one labeled p3m1 has rotational symmetry and three reflections, and all the centers of rotation lie on these axes. The one labeled p31m also has rotational symmetry and three axes of reflection, but some centers of rotation are not on these axes.

Wallpaper Patterns with Half-Turns and Reflections

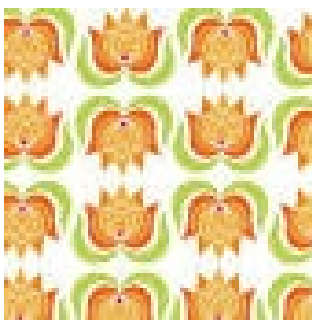


Table: pmm

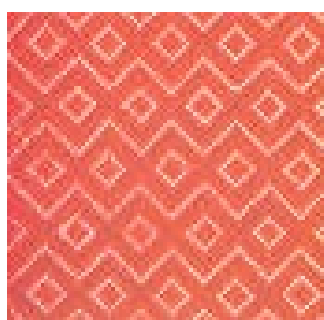


Table: cmm



Table: pmg

The pattern labeled pmm has reflections and trivial glide reflections. The pattern labeled cmm has reflections and a glide reflection along an axis parallel to a reflection axis. The pattern labeled pmg has reflections and a glide reflection along an axis not parallel to a reflection axis.

Wallpaper Patterns with Half-Turns, No Mirror Reflections



Table: pgg

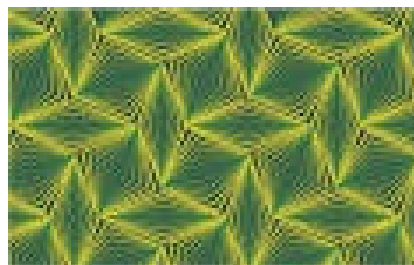


Table: p2

The pattern labeled p2 has only rotational symmetry: a half-turn. The pattern labeled pgg has a half-turn, no mirror reflections, but two glide reflections.

Wallpaper Patterns with No Rotational Symmetry



Table: p1: only translations



Table: pg: glide reflection

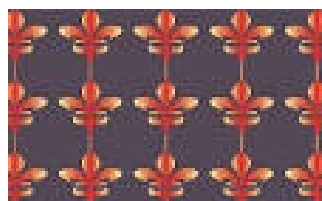


Table: pm: mirror reflection

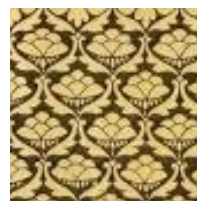


Table: cm: mirror and non-trivial glide reflections

Wallpaper Group Classification/Summary

No rotational symmetry:

1. $p1$: no reflections and no glide reflections.
2. pg : glide reflections, but no reflections.
3. pm : with reflections; any glide reflection axis is also a reflection axis.
4. cm : with reflections; some glide reflection axis is not a reflection axis.

With 2-fold rotations but no 4-fold rotation:

1. $p2$: no reflections and no glide reflections.
2. pgg : no reflections, with glide reflections.
3. pmm : with reflections; any glide reflection axis is also a reflection axis.
4. cmm : with reflections; some glide reflection axis is not a reflection axis but is parallel to a reflection axis.
5. pmg : with reflections; some glide reflection axis is not a reflection axis and is not parallel to any reflection axis.

With 4-fold rotations:

1. $p4$: no reflections.
2. $p4m$: with reflections; 4-fold rotation centers lie on reflection axes.
3. $p4g$: with reflections; 4-fold rotation centers do not lie on reflection axes.

With 3-fold rotations but no 6-fold rotation:

1. $p3$: no reflections.
2. $p3m1$: with reflections; any rotation center lies on a reflection axis.
3. $p31m$: with reflections; some rotation center does not lie on any reflection axis.

With 6-fold rotations:

1. $p6$: no reflections.
2. $p6m$: with reflections.