University of Toronto
SOLUTIONS to **MAT335H1F TERM TEST**
**Friday, October 19, 2012**
Duration: 50 minutes

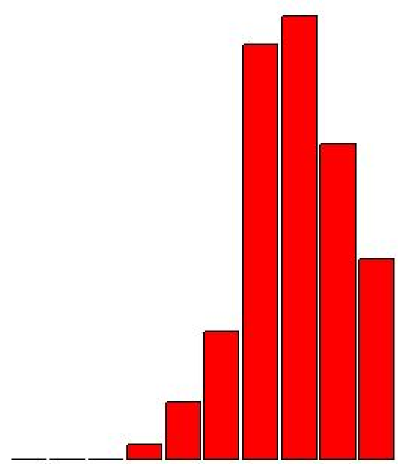
**Only aids permitted:** Scientific calculator, to be supplied by student.

**General Comments about the Test:**

- Question 1 was done perfectly many times, but Question 2 was done perfectly only once or twice.
- You have to give the restrictions on $c$ for both 2(a) and 2(b) to get full points.
- In 2(d), $F_1'(0) = 1$ and $F'_{-1}(0) = -1$, which are necessary conditions for a tangent node and a period-doubling bifurcation, respectively, but they are not sufficient conditions. So you still have to check what is happening to either side of the node.

**Breakdown of Results:** 110 students wrote this test. The marks ranged from 34% to 98%, and the average was 73.1%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

| Grade | % | Decade | % |
|---|---|---|---|
| | | 90-100% | 12.7% |
| A | 32.7% | 80-89% | 20.0% |
| B | 28.2% | 70-79% | 28.2% |
| C | 26.4% | 60-69% | 26.4% |
| D | 8.2% | 50-59% | 8.2% |
| F | 4.5% | 40-49% | 3.6% |
| | | 30-39% | 0.9% |
| | | 20-29% | 0.0% |
| | | 10-19% | 0.0% |
| | | 0-9% | 0.0% |



1. [25 marks; each part is worth 5 marks.] Let $T(x) = \begin{cases} 2x, & \text{if } x \le 1/2 \\ 2-2x, & \text{if } x > 1/2 \end{cases}$

(a) Find the fixed points of $T$.

**Solution:** if $x < 1/2$, then

$$T(x) = x \Leftrightarrow 2x = x \Leftrightarrow x = 0;$$

if $x > 1/2$, then

$$T(x) = x \Leftrightarrow 2 - 2x = x \Leftrightarrow x = 2/3.$$

So the only fixed points of $T$ are $x = 0$ and $x = 2/3$.

(b) Confirm that the orbit of $x_0 = 2/7$ under $T$ is periodic. What is the prime period?

**Solution:**

$$T(2/7) = \frac{4}{7};\ T(4/7) = 2 - \frac{8}{7} = \frac{6}{7};\ T(6/7) = 2 - \frac{12}{7} = \frac{2}{7}.$$

The prime period is 3.

(c) Explain why any periodic cycle of $T$ must be repelling.

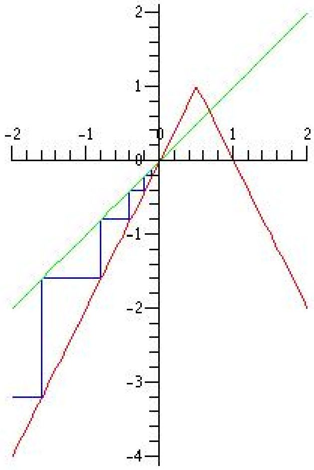
**Solution:** observe that for all $x \neq 1/2$, $|T'(x)| = 2$. (Since $x = 1/2$ is eventually fixed by $T$, you don't have to worry about $x = 1/2$ being on a periodic cycle of $T$.) Let $x_1, x_2, \ldots, x_n$ be an $n$-cycle of $T$. Then

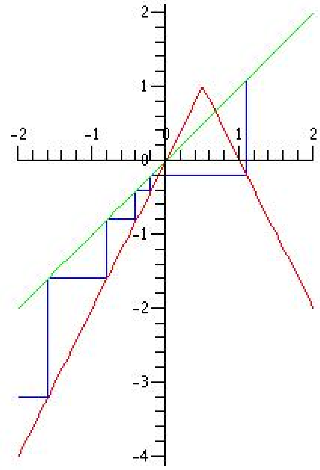
$$|T'(x_1)T'(x_2)\cdots T'(x_n)| = 2^n > 1,$$

so the $n$-cycle is repelling.

(d) What is the fate of the orbit of $x_0$ under $T$ if $x_0 < 0$ or $x_0 > 1$?

**Solution:** if $x_0 < 0$ or $x_0 > 1$, then $\lim_{n\to\infty} T^n(x_0) = -\infty$, as illustrated on the graphs below:



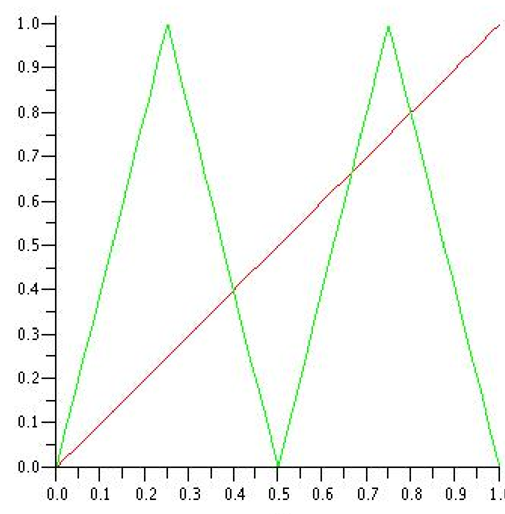




(e) Plot the graphs of $y = T^2(x)$ and $y = x$, for $0 \le x \le 1$. How many points of prime period 2 does $T$ have?

**Solution:** from the graph of $T^2$, shown to the right, you can see that it intersects the line

$$y = x$$

four times, but two of these intersection points are the fixed points of part (a). The other two points are the only period 2 points of $T$.



2. [25 marks] Consider the family of functions $F_c(x) = c\,x\,(1 - x^2)$.

(a) [5 marks] Find the fixed points of $F_c$.

**Solution:**

$$F_c(x) = x \Leftrightarrow cx(1 - x^2) = x \Leftrightarrow x = 0 \text{ or } x^2 = 1 - \frac{1}{c}.$$

So the fixed points are $x = 0$, and

$$x = \sqrt{\frac{c-1}{c}},\quad x = -\sqrt{\frac{c-1}{c}}$$

for $c < 0$ or $c \geq 1$.

(b) [5 marks] Solve the equation $F_c(x) = -x$ for $x$. What is a 2-cycle for $F_c$?

**Solution:** since $F_c$ is odd, solutions to $F_c(x) = -x$ will include period 2 points, as pointed out in class.

$$F_c(x) = -x \Leftrightarrow cx(1 - x^2) = -x \Leftrightarrow x = 0 \text{ or } x^2 = 1 + \frac{1}{c}.$$

Since $x = 0$ is a fixed point of $F_c$, the period 2 points are

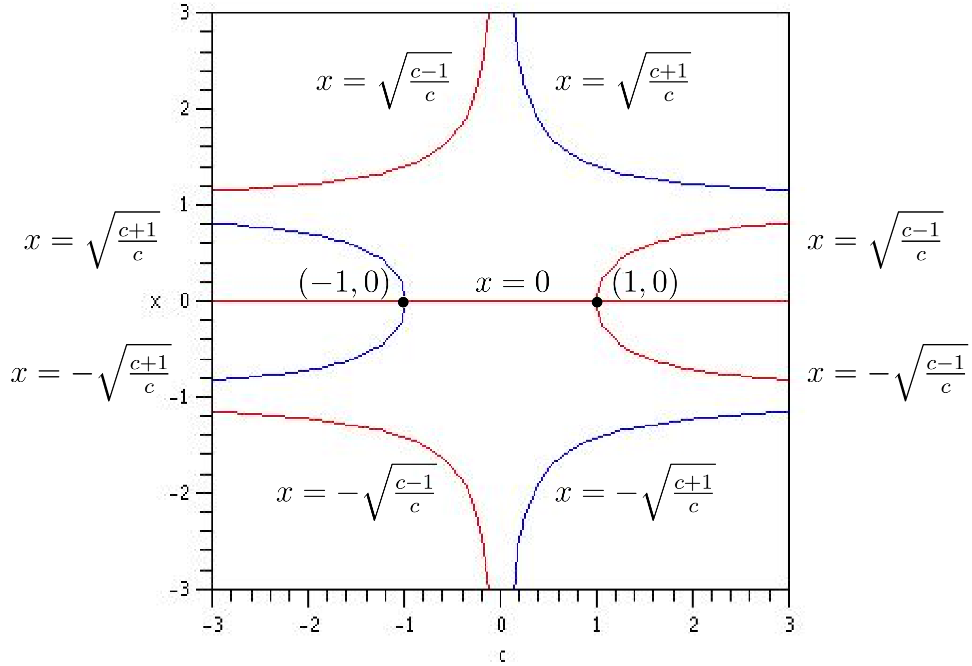
$$x = \sqrt{\frac{c+1}{c}},\quad x = -\sqrt{\frac{c+1}{c}}$$

for $c > 0$ or $c \leq -1$.

(c) [5 marks] Indicate your solutions from parts (a) and (b) on this diagram which represents (part of) the bifurcation diagram for $F_c(x)$, with

$$-3 \leq c \leq 3, -3 \leq x \leq 3.$$

**Solution:** the fixed points are in red, and the period 2 points are in blue.




(d) [10 marks] Classify the two nodes $(c, x) = (1, 0)$ and $(-1, 0)$ in the above diagram. That is, determine if each node is a saddle-node (or tangent bifurcation), a period-doubling bifurcation, or neither.

**Solution:** the node $(c, x) = (1, 0)$ is neither a saddle-node nor a period-doubling node, since all the lines on the bifurcation diagram connected to the node are fixed points.

The node $(c, x) = (-1, 0)$ is a period-doubling bifurcation since

1. the fixed point $x = 0$ is attracting for $c > -1$, neutral for $c = -1$ and repelling for $c < -1$, because $F_c'(0) = c$.
2. the 2-cycle
$$\sqrt{\frac{c+1}{c}},\ -\sqrt{\frac{c+1}{c}}$$
exists for $c \leq -1$, not for $c > -1$, and is attracting for $-2 < c < -1$, since
$$F_c'\left(\sqrt{\frac{c+1}{c}}\right) F_c'\left(-\sqrt{\frac{c+1}{c}}\right) = (c - 3(c+1))^2 = (-3 - 2c)^2,$$
so $F_c'\left(\sqrt{\frac{c+1}{c}}\right) F_c'\left(-\sqrt{\frac{c+1}{c}}\right) < 1 \Leftrightarrow -1 < 3+2c < 1 \Leftrightarrow -2 < c < -1.$