UNIVERSITY OF TORONTO
Faculty of Arts and Science
DECEMBER 2010 EXAMINATIONS

MAT335H1F Solutions

Chaos, Fractals and Dynamics
Examiner: D. Burbulla

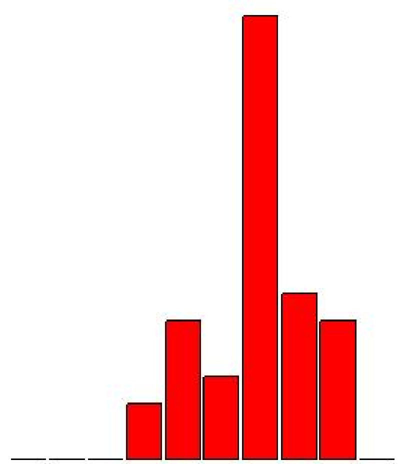
Duration - 3 hours
Examination Aids: A Scientific Hand Calculator

General Comments:

1. Very few students knew the definition of a chaotic dynamical system! (Question 3(a).)
2. Very few students used the conjugacy from part 1(f) to solve the parts of 1(g). Without conjugacy, 1(g) is extremely difficult!
3. Very few students made the connection between the negative Schwarzian derivative in part 1(i) and the orbit diagram in part 1(j).
4. The best mark on Question 1 was 48/50.
5. Nobody got Question 4 completely correct. Parts 4(b) and 4(c) were the hardest parts of the exam, but only counted for 7%.

Breakdown of Results: 37 students wrote this exam. The marks ranged from 37% to 89%, and the average was 64.2%. The average on Question 1 was 33.5/50 or 67%; the average on the rest of the exam was 30.7/50 or 61.4%. For most students the marks on the first and second half of the exam were very similar. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0 %
A	13.5%	80-89%	13.5%
B	16.2%	70-79%	16.2%
C	43.2%	60-69%	43.2%
D	8.1%	50-59%	8.1%
F	18.9%	40-49%	13.5%
		30-39%	5.4%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



1. [50 marks] This question has 10 parts. For the whole question let

$$F_\lambda(x) = x^2 + x + \lambda \;\text{ and }\; Q_c(x) = x^2 + c.$$

(a) [3 marks] Find the critical point of F_λ and the vertex of the parabola with equation $y = F_\lambda(x)$.

Solution: $F'_\lambda(x) = 2x + 1 = 0 \Leftrightarrow x = -1/2$; the vertex of the parabola is

$$\left(-\frac{1}{2}, \lambda - \frac{1}{4}\right).$$

(b) [5 marks] Find the fixed points of F_λ and determine for which values of λ they are repelling or attracting.

Solution:

$$F_\lambda(x) = x \Leftrightarrow x^2 + x + \lambda = x \Leftrightarrow x^2 + \lambda = 0 \Leftrightarrow x = \pm\sqrt{-\lambda}.$$

So for $\lambda < 0$ there are two fixed points: $x = \sqrt{-\lambda}$ and $x = -\sqrt{-\lambda}$.

$$|F'_\lambda(\sqrt{-\lambda})| = |2\sqrt{-\lambda} + 1| > 1 \text{ if } \lambda < 0;$$

so $x = \sqrt{-\lambda}$ is a repelling fixed point if $\lambda < 0$.

$$|F'_\lambda(-\sqrt{-\lambda})| = |-2\sqrt{-\lambda} + 1| < 1 \Leftrightarrow 1 > \sqrt{-\lambda} > 0 \Rightarrow -1 < \lambda < 0;$$

so $x = -\sqrt{-\lambda}$ is an attracting fixed point if $-1 < \lambda < 0$.

(c) [4 marks] Show that the family F_λ has a tangent bifurcation at $\lambda = 0$.

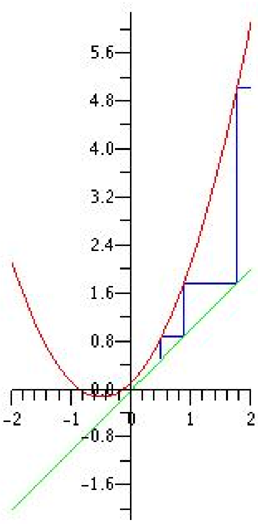
Solution: check the requirements for a tangent bifurcation.

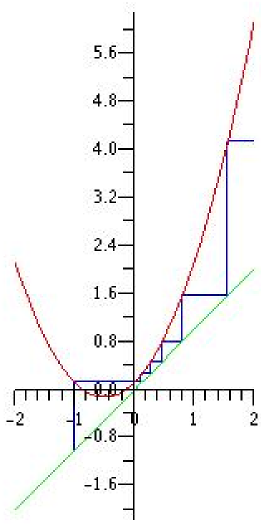
1. If $\lambda > 0$ then F_λ has no fixed points.
2. If $\lambda = 0$ then F_0 has a neutral fixed point at $x = 0$.
3. If $\lambda < 0$ then F_λ has two fixed points, $\pm\sqrt{-\lambda}$, one of which is repelling for $\lambda < 0$ and one of which is attracting for $-1 < \lambda < 0$.

(d) [4 marks] Use graphical analysis to show that if $\lambda > 0$ then for every $x \in \mathbb{R}$

$$\lim_{n\to\infty} F_\lambda^n(x) = \infty.$$

Solution: if $\lambda > 0$ the graph of $y = F_\lambda(x)$ is always above the graph of the line $y = x$. The following orbit diagrams illustrate the orbits of $x_0 = 0.5$ and $x_0 = -1$ under F_λ going to infinity, for $\lambda = 1/8$.



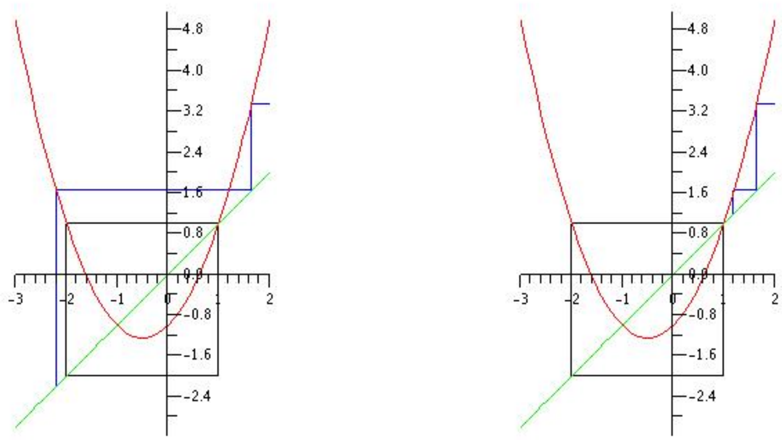


The divergence to infinity is more obvious the bigger $\lambda > 0$ is.

(e) [4 marks] Use graphical analysis to show that if $x \notin [-1-\sqrt{-\lambda}, \sqrt{-\lambda}]$ for $\lambda \le 0$ then

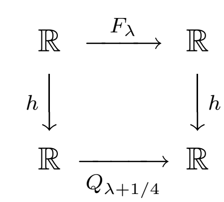
$$\lim_{n\to\infty} F_\lambda^n(x) = \infty.$$

Solution: $F_\lambda(-1-\sqrt{-\lambda}) = 1 + 2\sqrt{-\lambda} - \lambda - 1 - \sqrt{-\lambda} + \lambda = \sqrt{-\lambda}$, so both endpoints of $[-1-\sqrt{-\lambda}, \sqrt{-\lambda}]$ are eventually fixed. If x is not in this interval there are two cases, illustrated by the two graphs below, with $\lambda = -1$:



(f) [4 marks]

Check that $h(x) = x + 1/2$ is a conjugacy between F_λ and $Q_{\lambda+1/4}$. State clearly all the properties that h must satisfy.



Solution: You have to check that h is 1-1, onto, continuous, with continuous inverse—which are all obviously true—and that $h \circ F_\lambda = Q_{\lambda+1/4} \circ h$. This last equation should be verified:

$$(h \circ F_\lambda)(x) = h(F_\lambda(x)) = h(x^2 + x + \lambda) = x^2 + x + \lambda + \frac{1}{2};$$

and

$$(Q_{\lambda+1/4} \circ h)(x) = Q_{\lambda+1/4}(h(x)) = Q_{\lambda+1/4}(x + 1/2) = \left(x + \frac{1}{2}\right)^2 + \lambda + \frac{1}{4}$$

$$= x^2 + x + \lambda + \frac{1}{4} + \frac{1}{4} = x^2 + x + \lambda + \frac{1}{2}.$$

(g) [8 marks] Are the following statements True or False? Justify your answer.

(i) [4 marks] $F_{-9/4} : [-2.5, 1.5] \longrightarrow [-2.5, 1.5]$ is chaotic.

Solution: True. $h([-2.5, 1.5]) = [-2, 2]$ and

$$h \circ F_{-9/4} = Q_{-2} \circ h.$$

From class we know that Q_{-2} is chaotic on $[-2, 2]$; so by conjugacy, $F_{-9/4}$ is chaotic on $[-2.5, 1.5]$.

$$\begin{array}{ccc} [-2.5, 1.5] & \xrightarrow{F_{-9/4}} & [-2.5, 1.5] \\ h \big\downarrow & & \big\downarrow h \\ [-2, 2] & \xrightarrow[Q_{-2}]{} & [-2, 2] \end{array}$$

(ii) [4 marks] If $\lambda < -9/4$, then $\{x \in [-1 - \sqrt{-\lambda}, \sqrt{-\lambda}] \mid F_\lambda^n(x) \not\to \infty\}$ is a Cantor-like set.

Solution: True. Assume $\lambda < -9/4$, and let $J = [-1 - \sqrt{-\lambda}, \sqrt{-\lambda}]$, Then

$$h(J) = \left[-\frac{1}{2} - \sqrt{-\lambda}, \frac{1}{2} + \sqrt{-\lambda}\right]$$

and

$$Q_{\lambda+1/4}\left(\pm\left(\frac{1}{2} + \sqrt{-\lambda}\right)\right) = \frac{1}{4} + \sqrt{-\lambda} - \lambda + \lambda + \frac{1}{4} = \frac{1}{2} + \sqrt{-\lambda}.$$

That is

$$h(J) = [-p_+, p_+] = I,$$

where

$$p_+ = \frac{1}{2} + \sqrt{-\lambda}$$

is the positive fixed point of $Q_{\lambda+1/4}$. Also: $\lambda < -9/4 \Rightarrow \lambda + 1/4 < -2$. Let

$$K = \{x \in J \mid F_\lambda^n(x) \not\to \infty\} \text{ and } \Lambda = \left\{x \in I \mid Q_{\lambda+1/4}^n(x) \not\to \infty\right\}.$$

Since Λ is a Cantor-like set, as covered in class, K too is a Cantor-like set, by conjugacy.

$$\begin{array}{ccc} K & \xrightarrow{F_\lambda} & K \\ h \big\downarrow & & \big\downarrow h \\ \Lambda & \xrightarrow[Q_{\lambda+1/4}]{} & \Lambda \end{array}$$

(h) [10 marks] Find both points of prime period 2 for F_λ and determine for which values of λ the 2-cycle is repelling or attracting. What kind of bifurcation, if any, does F_λ have at $\lambda = -1$? Draw the bifurcation diagram of F_λ, for $-3/2 \le \lambda \le 1/2$.

Solution:

$$\begin{aligned} F_\lambda^2(x) = x \quad &\Rightarrow \quad (x^2 + x + \lambda)^2 + x^2 + x + \lambda + \lambda = x \\ &\Rightarrow \quad x^4 + 2x^3 + 2(1+\lambda)x^2 + 2\lambda x + \lambda^2 + 2\lambda = 0 \\ &\Rightarrow \quad (x^2 + \lambda)(x^2 + 2x + 2 + \lambda) = 0 \\ &\Rightarrow \quad x^2 = -\lambda \text{ or } x = \frac{-2 \pm \sqrt{4 - 4\,(2+\lambda)}}{2} \\ &\Rightarrow \quad x = \pm\sqrt{-\lambda} \text{ or } x = -1 \pm \sqrt{-1-\lambda} \end{aligned}$$

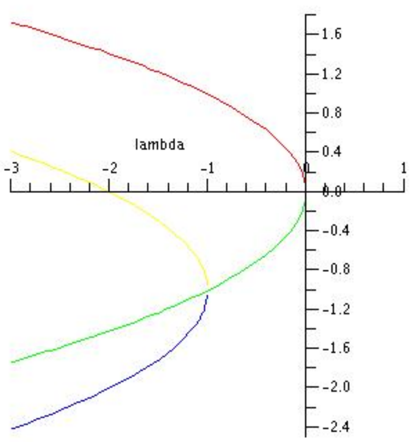
$F'_\lambda(x) = 2x + 1$, so

$$F'_\lambda(-1-\sqrt{-1-\lambda})F'_\lambda(-1+\sqrt{-1-\lambda}) = (-1-2\sqrt{-1-\lambda})(-1+2\sqrt{-1-\lambda}) = 5+4\lambda$$

and the 2-cycle is attracting if and only if

$$|5 + 4\lambda| < 1 \Leftrightarrow -1 < 5 + 4\lambda < 1 \Leftrightarrow -\frac{3}{2} < \lambda < -1.$$

Bifurcation Diagram: there is a period-doubling bifurcation at $\lambda = -1$.



[red] $x = \sqrt{-\lambda}$ is a repelling fixed point for all $\lambda < 0$

[green] $x = -\sqrt{-\lambda}$ is an attracting fixed point for $-1 < \lambda < 0$.

[yellow] $x = -1 + \sqrt{-1-\lambda}$ is an attracting point of period 2 for $-1.5 < \lambda < -1$.

[blue] $x = -1 - \sqrt{-1-\lambda}$ is an attracting point of period 2 for $-1.5 < \lambda < -1$.

(i) [3 marks] Compute $S(F_\lambda)(x)$, the Schwarzian derivative of F_λ.

Solution: $F'_\lambda(x) = 2x + 1,\ F''_\lambda(x) = 2,\ F'''_\lambda(x) = 0$, so

$$S(F_\lambda)(x) = \frac{F'''_\lambda(x)}{F'_\lambda(x)} - \frac{3}{2}\left(\frac{F''_\lambda(x)}{F'_\lambda(x)}\right)^2 = 0 - \frac{3}{2}\left(\frac{2}{2x+1}\right)^2 = -\frac{6}{(2x+1)^2} < 0.$$

(j) [5 marks] Explain why the orbit diagram for the family F_λ, $-9/4 \le \lambda \le 0$, can be obtained by considering all orbits of $x = -1/2$ under F_λ.

Solution: since $S(F_\lambda)(x) < 0$, the basin of attraction for any periodic point of F_λ will extend to infinity or include a critical point of F_λ. By part (e) the basin of attraction cannot extend to infinity, so the basin of attraction must include a critical point of F_λ. The only critical point of F_λ is $x = -1/2$; thus the orbit diagram for the family F_λ can be obtained by considering all the orbits of $x = -1/2$ under F_λ.

(The significance of the restriction $-9/4 \le \lambda \le 0$ is that then

$$F_\lambda : [-2.5, 1.5] \longrightarrow [-2.5, 1.5],$$

as shown in part (g)*(i)*. Consequently all orbits of $x = -1/2$ are bounded for $-9/4 \le \lambda \le 0$. Of course, the orbit diagram of the family F_λ for $-9/4 \le \lambda \le 0$ is just the orbit diagram of the family

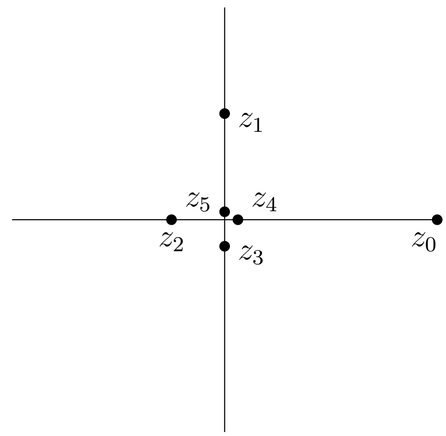
$$Q_c : [-2, 2] \longrightarrow [-2, 2]$$

for $-2 \le c \le 0.25$, appropriately shifted.)

2.(a) [4 marks] Sketch the orbit of 1 in the complex plane under $L_\alpha(z) = \alpha z$, if $\alpha = i/2$.

Solution: let $z_n = L_\alpha^n(1)$. Then

$$
\begin{aligned}
z_0 = 1 \quad &\Rightarrow \quad z_1 = \frac{i}{2} \\
&\Rightarrow \quad z_2 = -\frac{1}{4} \\
&\Rightarrow \quad z_3 = -\frac{i}{8} \\
&\Rightarrow \quad z_4 = \frac{1}{16} \\
&\Rightarrow \quad z_5 = \frac{i}{32} \\
\text{etc} \quad &\quad \lim_{n\to\infty} z_n = 0
\end{aligned}
$$



2.(b) [6 marks] Show that $z_0 = e^{2\pi i/7}$ lies on a cycle of period 3 for $Q_0(z) = z^2$. Is this cycle attracting, repelling or neutral?

Solution: if $z = e^{i\theta}$ then $Q_0(z) = z^2 = e^{2i\theta}$, so

$$z_1 = Q_0(z_0) = e^{4\pi i/7};\ z_2 = Q_0(z_1) = e^{8\pi i/7};\ z_3 = Q_0(z_2) = e^{16\pi i/7} = e^{2\pi i/7} = z_0$$

since $e^{14\pi i/7} = 1$. The 3-cycle is repelling since $Q_0'(z) = 2z$ and

$$|Q'(z_0)||Q'(z_1)||Q_0'(z_2)| = 2 \cdot 2 \cdot 2 = 8 > 1.$$

3.(a) [5 marks] According to Devaney, a dynamical system $F : X \longrightarrow X$ is chaotic if it has three properties:

1. Density

   Solution: the periodic points of F are dense in X.

2. Transitivity

   Solution: if $x, y \in X$ and $\epsilon > 0$ is given, then there is $z \in X$ such that $d[x, z] < \epsilon$ and $d[F^k(z), y] < \epsilon$ for some k.

3. Sensitivity

   Solution: there is a $\beta > 0$ such that for any $x \in X$ and any $\epsilon > 0$ there is a $y \in X$ such that $d[x, y] < \epsilon$ and $d[F^k(x), F^k(y)] \geq \beta$, for some k.

Define precisely, in terms of F and X, what each of these properties is.

3.(b) [5 marks] Prove that the orbit of

$$\hat{\mathbf{s}} = (\underbrace{0\,1}_{\text{all 1 blocks}}\ \underbrace{00\,01\,10\,11}_{\text{all 2 blocks}}\ \underbrace{000\,010\,101\,\ldots}_{\text{all 3 blocks}}\ \ldots)$$

under the shift map σ is dense in Σ.

Solution: let $\mathbf{t} = (t_0 t_1 t_2 \ldots t_n \ldots)$ be an arbitrary sequence in Σ and let $\epsilon > 0$ be given. Pick n large enough so that

$$\frac{1}{2^n} < \epsilon.$$

Pick k large enough such that the first $n + 1$ entries of $\sigma^k(\hat{\mathbf{s}})$ form the $(n + 1)$-block $t_0 t_1 t_2 \ldots t_n$. Then

$$d[\sigma^k(\hat{\mathbf{s}}), \mathbf{t}] \leq \frac{1}{2^n} < \epsilon.$$

4. [10 marks] Suppose $F : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

(a) [3 marks] Explain why F must have a point of prime period 56 if it has a point of prime period 60.

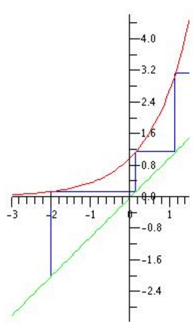
Solution: $56 = 2^3 \cdot 7$ and $60 = 2^2 \cdot 15$, so 60 is before 56 in the Sarkovskii ordering. By Sarkovskii's Theorem, F must have a point of prime period 56 if it has a point of prime period 60.

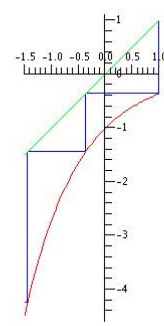
(b) [4 marks] If F is increasing for all x explain why

1. all orbits of x under F are unbounded, ie. $|F^n(x)| \to \infty$,
2. or else F has a fixed point, but no periodic points of any prime period greater than 1.

Solution:

If the graph of $F(x)$ doesn't intersect the line $y = x$ then the orbit of x under F is unbounded, as illustrated by the two graphic analyses to the right:





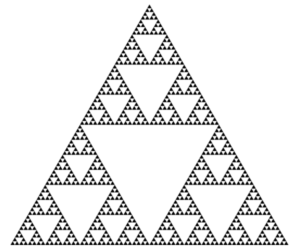
Otherwise the graph of $F(x)$ intersects the line $y = x$ and F has at least one fixed point of period 1. Suppose $F^2(x) = x$. If $F(x) < x$, then $F^2(x) < F(x) < x$, since F is increasing. Similarly, if $x < F(x)$, then $x < F(x) < F^2(x)$. In each case we contradict the fact that $F^2(x) = x$. The only possibility left is that $F(x) = x$, so x is actually a fixed point, not a point of prime period 2. Finally, by Sarkovskii's Theorem, F has no periodic point with prime period greater than 1, since 2 is the second last number in the Sarkovskii ordering.

(c) [3 marks] If F is decreasing for all x explain why F has a fixed point, but no periodic points of any prime period greater than 2.

Solution: F is decreasing for all x so its graph must intersect the line $y = x$. Now, let $G(x) = F^2(x)$. Since $G'(x) = F'(F(x))\,F'(x) > 0$, G^2 is increasing with at least one fixed point. By part (b), $G^2(x) = x \Rightarrow G(x) = x$; that is, $F^4(x) = x \Rightarrow F^2(x) = x$. So F has no periodic point with prime period 4. By Sarkovskii's Theorem F has no periodic points with prime period greater than 2, since 4 is the third last number in the Sarkovskii ordering.

5. [10 marks]

Describe clearly and briefly two different algorithms that both generate the Sierpinski triangle. What is the fractal dimension of the Sierpinski triangle?



Solution: we mentioned three algorithms in class, which are really three 'sides' of the same coin. Any two of them will do.

Algorithm 1: the chaos game. Start with an equilateral triangle with vertices A, B, C. Pick a point p_0 anywhere in the plane. Randomly pick one of the three vertices and go half way from p_0 to the selected vertex, giving you p_1. Randomly pick one of the three vertices and go halfway from p_1 to the selected vertex, giving you p_2. Repeat this process ad infinitum. The attractor of this infinite process is the Sierpinski triangle.

Algorithm 2: an Iterated Function System. This is a formalization of the first one. Let

$$A_1 \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x \\ y \end{bmatrix}; A_2 \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x-1 \\ y \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}; A_3 \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x - \frac{1}{2} \\ y - \frac{\sqrt{3}}{2} \end{bmatrix} + \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix};$$

then the fixed points of A_1, A_2, A_3 are the vertices of an equilateral triangle. Now pick an arbitrary initial point and compute its orbit under random iterations of A_1, A_2, A_3. The orbit will converge to the Sierpinski triangle (with vertices the fixed points of A_1, A_2, A_3.)

Algorithm 3: the removal algorithm.

Start with an equilateral triangle. Remove the middle equilateral triangle. In each of the three remaining equilateral triangles, remove the middle equilateral triangle. In each of the subsequent 9 equilateral triangles, remove the middle equilateral triangle. Repeat this process ad infinitum. What's left is the Sierpinski triangle.

Fractal Dimension:

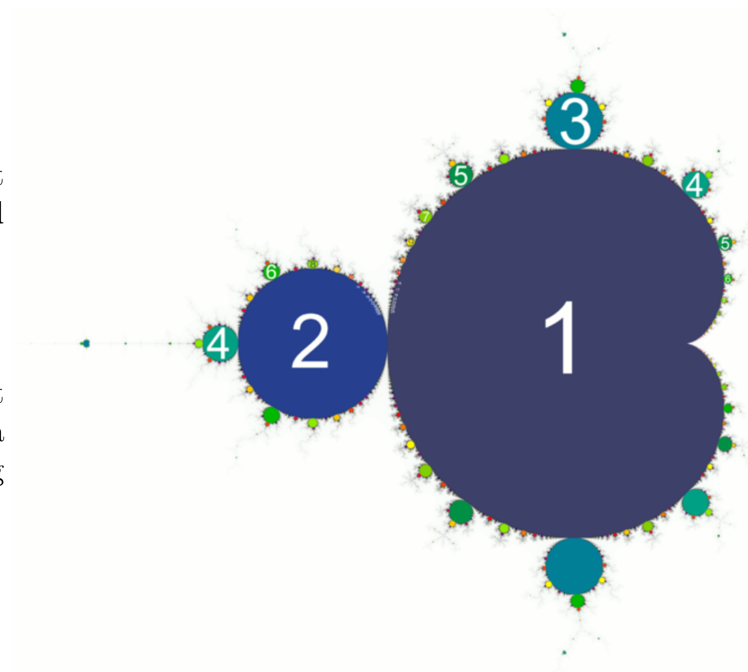
$$D = \frac{\log 3}{\log 2} \simeq 1.585$$

6. [10 marks]

To the right is the Mandelbrot set, with some of its period bulbs labeled. Let

$$Q_c(z) = z^2 + c;$$

let K_c be the filled Julia set of Q_c; and let J_c be the Julia set of Q_c. Answer the following five questions, 2 marks each:



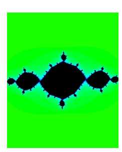
(a) In general, what is the connection between K_c and J_c?

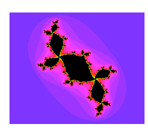
Solution: J_c is the boundary of K_c.

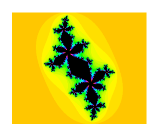
(b) If c is in the period 2 bulb what is the eventual fate of the orbit of 0 under Q_c?

Solution: the orbit of 0 under Q_c is eventually attracted to a 2-cycle.

(c) If c is in a period 3 bulb, which one of the following could be K_c?







Solution: the middle fractal, which has 3-fold similarity.

(d) If $c = -1.8 + 1.8i$, is K_c connected or totally disconnected?

Solution: c is not in the Mandlebrot set, so K_c is totally disconnected.

(e) Above the Mandelbrot set, and properly aligned with it, sketch the orbit diagram of $Q_c : \mathbb{R} \longrightarrow \mathbb{R}$ for $-2 \leq c \leq 0.25$.

Solution: