UNIVERSITY OF TORONTO
Faculty of Arts and Science
Solutions to **FINAL EXAMINATION, DECEMBER 2016**
MAT301H1F: Groups and Symmetry
Examiner: D. Burbulla

Duration - 3 hours

NAME: ______________________________ STUDENT NUMBER: ____________________

INSTRUCTIONS: Present your solutions to the following questions in the space provided. No aids of any kind permitted. Do not tear any pages from this exam. The marks for each question are indicated in parentheses beside the question number. **TOTAL MARKS:** 100

Notation: If $n \geq 2$, $\mathbb{Z}_n$ is the additive group of integers modulo n, $U(n)$ is the multiplicative group of units modulo n consisting of all positive integers less than n and relatively prime to n, C_n is the cyclic group of order n, S_n is the symmetric group of degree n, A_n is the alternating group of degree n, $GL(n, \mathbb{R})$ is the group of $n \times n$ invertible matrices with real entries; if $n \geq 3$, D_n is the dihedral group of order $2n$. If $G_1, G_2, \ldots, G_k$ are groups, $G_1 \oplus G_2 \oplus \cdots \oplus G_k$ is the external direct product of $G_1, G_2, \ldots, G_k$.

Recall: If A is an orthogonal 3×3 matrix[1] then the following chart can be used to classify the geometry of the transformation represented by A, depending on $\det(A)$ and $\operatorname{tr}(A)$.

$\det(A)$	$\operatorname{tr}(A)$	action of A	order of A	$\det(A)$	$\operatorname{tr}(A)$	action of A	order of A
1	3	identity, $A = I$	1	-1	-3	inversion, $A = -I$	2
1	-1	rotation	2	-1	1	reflection	2
1	0	rotation	3	-1	0	improper rotation	6
1	1	rotation	4	-1	-1	improper rotation	4
1	ψ	rotation	5	-1	$-\psi$	improper rotation	10
1	ϕ	rotation	5	-1	$-\phi$	improper rotation	10
1	2	rotation	6	-1	-2	improper rotation	6

where

$$\phi = \frac{1+\sqrt{5}}{2} \text{ and } \psi = \frac{1-\sqrt{5}}{2}.$$

[1] That is, $A^{-1} = A^T$

1. [10 marks] Define the following:

   (a) [2 marks] H is a **normal subgroup** of the group G, given that $H \leq G$.

   Definition: H is a normal subgroup of G if $gH = Hg$ for all $g \in G$.

   (b) [2 marks] a p**-group**, for p a prime.

   Definition: G is a p-group if $|G| = p^k$, for some positive integer $k \geq 1$.

   (c) [2 marks] two elements a and b are **conjugate** in G.

   Definition: $a, b \in G$ are conjugate if there is a $g \in G$ such $b = gag^{-1}$.

   (d) [2 marks] an **inner automorphism** of a group G.

   Definition: for any $g \in G$ the mapping $\phi_g : G \longrightarrow G$ defined by

   $$\phi_g(x) = gxg^{-1}$$

   is an inner automorphism of G.

   (e) [2 marks] an **even permutation** in S_n.

   Definition: $\sigma \in S_n$ is an even permutation if it can be written as the product of an even number of 2-cycles (or transpositions) in S_n.

2. [20 marks] No justifications for your answers to the following questions are required:

(a) [5 marks] Consider the wall paper pattern exhibited to the right. Which of the following symmetries does the pattern have?

(*i*) a rotation of order 2	Yes
(*ii*) a rotation of order 3	No
(*iii*) a rotation of order 4	Yes
(*iv*) a rotation of order 5	No
(*v*) a rotation of order 6	No
(*vi*) a horizontal reflection	Yes
(*vii*) a vertical reflection	Yes
(*viii*) a diagonal reflection	No
(*ix*) a glide reflection	Yes

(b) [6 marks] List all possible non-isomorphic Abelian groups of order $504 = 8 \times 9 \times 7$.

Solution: use the Fundamental Theorem of Abelian Groups. There are six:

$$\mathbb{Z}_8 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7,\ \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7,$$

$$\mathbb{Z}_8 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7,\ \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7.$$

(c) [5 marks] List all five non-isomorphic groups of order 8.

Solution:

$$\mathbb{Z}_8,\ \mathbb{Z}_4 \oplus \mathbb{Z}_2,\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2,\ D_4,\ \mathcal{Q},$$

where $\mathcal{Q}$ represents the quaternions.

(d) [4 marks] Express $U(165)$ as the internal direct product of two proper subgroups in three different ways; **and** express $U(165)$ as the external direct product of three groups of the form $U(p)$, for p prime. (Note: $165 = 3 \times 5 \times 11$.)

Solution:

$$U(165) = U_3(165) \times U_{55}(165) = U_{15}(165) \times U_{11}(165) = U_{33}(165) \times U_5(165),$$

where

$$U_k(165) = \{x \in U(165) \mid x \equiv 1 \bmod k\}.$$

Also,

$$U(165) \approx U(3) \oplus U(5) \oplus U(11).$$

3. [10 marks] The parts of this question are unrelated.

(a) [5 marks] Find the order of $(5,1) + \langle(4,2)\rangle$ in the factor group $\mathbb{Z}_8 \oplus \mathbb{Z}_3/\langle(4,2)\rangle$.

Solution: since the order of $(4,2)$ in $\mathbb{Z}_8 \oplus \mathbb{Z}_3$ is $2 \times 3 = 6$,

$$|\mathbb{Z}_8 \oplus \mathbb{Z}_3/\langle(4,2)\rangle| = \frac{8 \times 3}{6} = 4.$$

Thus the order of $(5,1) + \langle(4,2)\rangle$ is 1, 2 or 4. We have

$$\langle(4,2)\rangle = \{(4,2),(0,1),(4,0),(0,2),(4,1),(0,0)\}.$$

- Since $(5,1) \notin \langle(4,2)\rangle$, the order of $(5,1) + \langle(4,2)\rangle$ is not 1.
- Since $2(5,1) \equiv (2,2) \notin \langle(4,2)\rangle$, the order of $(5,1) + \langle(4,2)\rangle$ is not 2.
- Thus the order of $(5,1) + \langle(4,2)\rangle$ must be 4, which can be verified directly:

$$4(5,1) \equiv (4,1) \in \langle(4,2)\rangle.$$

(b) [5 marks] Show that $\text{Aut}\,(\mathbb{Z}_{33}) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{10}$.

Solution: use the facts that

1. $\text{Aut}\,(\mathbb{Z}_n) \approx U(n)$,
2. $U(p\,q) \approx U(p) \oplus U(q)$, if p and q are relatively prime,
3. $U(p) \approx \mathbb{Z}_{p-1}$, if p is prime.

So

$$\text{Aut}\,(\mathbb{Z}_{33}) \approx U(33) = U(3 \times 11) \approx U(3) \oplus U(11) \approx \mathbb{Z}_2 \oplus \mathbb{Z}_{10}.$$

4. [10 marks] Suppose $(G, \cdot)$ is a multiplicative group with identity element e. Pick a in G, such that $a \neq e$, and define a new multiplication $*$ on G by

$$x * y = x \cdot a^{-1} \cdot y$$

(a) [7 marks] Prove that $(G, *)$ is also a group.

Solution: let x, y, z be elements of G.
Associativity: need to show $x * (y * z) = (x * y) * z$:

$$x * (y * z) = x * (y \cdot a^{-1} \cdot z) = x \cdot a^{-1} \cdot (y \cdot a^{-1} \cdot z) = x \cdot a^{-1} \cdot y \cdot a^{-1} \cdot z$$

$$(x * y) * z = (x \cdot a^{-1} \cdot y) * z = (x \cdot a^{-1} \cdot y) \cdot a^{-1} \cdot z = x \cdot a^{-1} \cdot y \cdot a^{-1} \cdot z$$

Existence of Identity: claim a is the identity element in $(G, *)$:

$$x * a = x \cdot a^{-1} \cdot a = x \cdot e = x \text{ and } a * x = a \cdot a^{-1} \cdot x = e \cdot x = x$$

Existence of Inverses: claim $a \cdot x^{-1} \cdot a$ is the inverse of x in $(G, *)$:

$$x * (a \cdot x^{-1} \cdot a) = x \cdot a^{-1} \cdot (a \cdot x^{-1} \cdot a) = x \cdot (a^{-1} \cdot a) \cdot x^{-1} \cdot a = (x \cdot x^{-1}) \cdot a = a$$

$$(a \cdot x^{-1} \cdot a) * x = (a \cdot x^{-1} \cdot a) \cdot a^{-1} \cdot x = a \cdot x^{-1} \cdot (a \cdot a^{-1}) \cdot x = a \cdot (x^{-1} \cdot x) = a$$

(b) [3 marks] Are $(G, \cdot)$ and $(G, *)$ isomorphic? Justify your answer.

Solution: yes, they are isomorphic. Define $f : (G, \cdot) \longrightarrow (G, *)$ by $f(x) = a \cdot x$. Then

$$f(x \cdot y) = a \cdot (x \cdot y) = a \cdot x \cdot y$$

and

$$f(x) * f(y) = (a \cdot x) * (a \cdot y) = (a \cdot x) \cdot a^{-1} \cdot (a \cdot y) = a \cdot x \cdot (a^{-1} \cdot a) \cdot y = a \cdot x \cdot y.$$

So f is a homomorphism. We need to show it is also one-to-one and onto:

$$f(x) = a \Rightarrow a \cdot x = a \Rightarrow x = e; \text{ so } \ker(f) = \{e\} \text{ and } f \text{ is one-to-one;}$$

$$\text{for } y \in (G, *), \; f(a^{-1} \cdot y) = a \cdot (a^{-1} \cdot y) = y, \text{ so } f \text{ is onto.}$$

Thus f is an isomorphism and so $(G, \cdot)$ is isomorphic to $(G, *)$.

5 [10 marks] Recall that $D_5 = \langle a, b \mid a^5 = b^2 = e, bab = a^4 \rangle$.

(a) [3 marks] How many homomorphisms are there from D_5 onto $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?

Solution: None. If $f : D_5 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\text{im}\,(f) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then $|\text{im}\,(f)| = 4$, but 4 does not divide $|D_5| = 10$.

(b) [7 marks] How many homomorphisms f are there from D_5 to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?

Solution: Let $f : D_5 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be a homomorphism. Then $|\text{im}\,(f)|$ divides both 4 and 10, so $|\text{im}\,(f)| = 1$ or 2. Consequently, by the First Isomorphism Theorem, $|\ker(f)| = 10$ or 5, respectively.

Case 1: $|\ker(f)| = 10$ and $|\text{im}\,(f)| = 1$. Then the only possibility is

$$f : D_5 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

by $f(x) = (0, 0)$, for all $x \in D_5$.

Case 2: $|\ker(f)| = 5$ and $|\text{im}\,(f)| = 2$.

Any subgroup of D_5 with order 5 will be normal, since $10/5 = 2$. But such a normal subgroup cannot contain a reflection, since 2 does not divide 5. Thus the only choice for the kernel of f is

$$\ker(f) = \langle a \rangle = \{e, a, a^2, a^3, a^4\}$$

and we mast have $f(a) = (0, 0)$. Finally, what are the possibilities for $f(b)$? Since $|\text{im}\,(f)| = 2$, the order of $f(b)$ must be 2. Thus

$$f(b) = (1, 0), \quad (0, 1) \text{ or } (1, 1),$$

giving three possibilities for f.

Conclusion: altogether there are four homomorphisms from D_5 to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

6. [10 marks] For the group S_4 find the following:

(a) [6 marks] the number of elements in each of its conjugacy classes.

Solution: in S_4 elements are conjugates if and only if they have the same cycle structure.

Cycle structure	Number of conjugates	Order	Parity
(1)	1	1	even
(ab)	$(4 \times 3)/2 = 6$	2	odd
(abc)	$(4 \times 3 \times 2)/3 = 8$	3	even
$(abcd)$	$(4 \times 3 \times 2 \times 1)/4 = 6$	4	odd
$(ab)(cd)$	$\frac{1}{2}\left(\frac{4 \times 3}{2} \times \frac{2 \times 1}{2}\right) = 3$	2	even

So

$$|\mathrm{cl}\,((1))| = 1,\ |\mathrm{cl}\,((12))| = 6,\ |\mathrm{cl}\,((123))| = 8, |\mathrm{cl}\,((1234))| = 6, |\mathrm{cl}\,((12)(34))| = 3.$$

(b) [4 marks] the number of its 3-Sylow subgroups, and the number of its 2-Sylow subgroups.

Solution: $|S_4| = 24 = 3 \times 8$. Each 3-Sylow subgroup has three elements, two of which have order 3, and there are eight elements of order 3 in S_4, so there must be **four** 3-Sylow subgroups in S_4.

Each 2-Sylow subgroup has $2^3 = 8$ elements, and all 15 elements of even order in S_4 must be in a 2-Sylow subgroup. Hence there must be more than one 2-Sylow subgroup. If $n \neq 1$ is the number of 2-Sylow subgroups in S_4, then

$$n \equiv 1 \bmod 2 \text{ and } n \text{ divides } |S_4| = 24,$$

whence $n = 3$.

Aside: the three 2-Sylow subgroups of S_4 are all isomorphic to D_4. You can picture this if you consider S_4 as the rotational symmetries of a cube, and you consider the rotations that fix each face—each pair of opposite faces, actually.

7. [8 marks] Suppose that G is a non-Abelian finite group,

$$\phi : G \longrightarrow \mathbb{Z}_{50} \oplus \mathbb{Z}_6$$

is a homomorphism, and ϕ is onto. Show that G must have an element of order 75.

Solution: The order of $(2, 2) \in \mathbb{Z}_{50} \oplus \mathbb{Z}_6$ is $\text{lcm}(25, 3) = 75$. Since ϕ is onto, there is an element $x \in G$ such that $\phi(x) = (2, 2)$, and so $\phi(x)$ has order 75 in $\mathbb{Z}_{50} \oplus \mathbb{Z}_6$. Then

$$|\phi(x)| \text{ divides } |x| \Rightarrow |x| = 75\,k,$$

for some positive integer k. Then x^k is an element in G with order 75.

Alternate Solution: by the First Isomorphism Theorem,

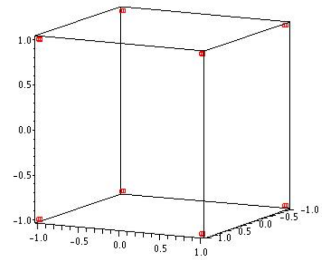
$$G/\ker(\phi) \approx \text{im}\,(\phi) = \mathbb{Z}_{50} \oplus \mathbb{Z}_6.$$

Thus there is an element $x \in G$ such that $x \ker(\phi)$ has order 75 in $G/\ker(\phi)$. Let the order of x in G be n. Then

$$\begin{aligned} x^n = e \quad &\Rightarrow \quad (x \ker(\phi))^n = x^n \ker(\phi) = e \ker(\phi) = \ker(\phi) \\ &\Rightarrow \quad 75 \text{ divides } n, \text{ since the order of } x \ker(\phi) \text{ is } 75 \\ &\Rightarrow \quad |x| = 75k, \text{ for some positive integer } k \geq 1 \\ &\Rightarrow \quad |x^k| = 75 \end{aligned}$$

So x^k is an element in G with order 75.

8. [22 marks] Consider the cube in the figure to the right, with vertices $(\pm 1, \pm 1, \pm 1)$, with respect to the usual x, y and z axes, and centre at the origin $(0, 0, 0)$. Let $S(C)$ be the symmetry group of this cube. Recall that $S(C)$ consists of the following 48 matrices, with usual matrix multiplication:



$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm 1 \\ \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \end{bmatrix},$$

$$\begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm 1 \\ 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \pm 1 & 0 \\ \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}.$$

Let $A = (a_{ij})$ be a matrix in $S(C)$ and define $\Phi : S(C) \longrightarrow S(C)$ by $\Phi\left((a_{ij})\right) = \left(a_{ij}^2\right)$. That is, the effect of Φ on a matrix A is to square *each entry* of A.

(a) [3 marks] Prove that Φ is a homomorphism. (Even if you can't prove part (a), you can use it for subsequent parts of this question.)

Solution: let $A = (a_{ij}),\ B = (b_{ij})$. Then $AB = (a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j})$ and

$$\begin{aligned} \Phi(AB) = \Phi(A)\Phi(B) \quad &\Leftrightarrow \quad \left((a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j})^2\right) = \left(a_{i1}^2 b_{1j}^2 + a_{i2}^2 b_{2j}^2 + a_{i3}^2 b_{3j}^2\right) \\ &\Leftrightarrow \quad (a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j})^2 = a_{i1}^2 b_{1j}^2 + a_{i2}^2 b_{2j}^2 + a_{i3}^2 b_{3j}^2 \\ &\Leftrightarrow \quad a_{i1}a_{i2}b_{1j}b_{2j} + a_{1i}a_{i3}b_{1j}b_{3j} + a_{i2}a_{i3}b_{2j}b_{3j} = 0 \end{aligned}$$

which last statement is true since for any matrix A or B in $S(C)$ every row of A has two zeros, as does every column of B.

(b) [6 marks] Find the kernel and image of Φ, and identify them: to which groups are they isomorphic? (At least, give their order.)

Solution:

$$\ker(\Phi) = \{A \in S(C) \mid \Phi(A) = I\} = \left\{ \begin{bmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{bmatrix} \right\},$$

which is a group isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and has 8 elements.

The image of Φ consists of all the matrices in $S(C)$ that have no negative entries, that is the 6 matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

These 6 matrices represent the identity, two rotations of order 3, and three reflections. So $\mathrm{im}\,(\Phi)$ is isomorphic to D_3.

OR: each of these 6 matrices is a permutation of the rows (or columns) of the 3×3 identity matrix I, so $\mathrm{im}\,(\Phi)$ is also isomorphic to S_3.

(c) [4 marks] Recall that $\det : S(C) \longrightarrow \{1, -1\}$, where $\det(A)$ is the determinant of the 3×3 matrix A and $\{1, -1\}$ is a group with multiplication, is a homomorphism. Show that both $p : S(C) \longrightarrow \{1, -1\}$ and $q : S(C) \longrightarrow \{1, -1\}$ defined by

$$p(A) = \det(\Phi(A)) \text{ and } q(A) = \det(A\,\Phi(A))$$

are also homomorphisms.

Solution: use properties of determinants and part (a). Watch your brackets!

$$p(AB) = \det(\Phi(AB)) = \det(\Phi(A)\Phi(B)) = \det(\Phi(A))\,\det(\Phi(B)) = p(A)\,p(B);$$

similarly,

$$\begin{aligned} q(AB) &= \det(AB\,\Phi(AB)) \\ &= \det(AB\,\Phi(A)\Phi(B)) \\ &= \det(A)\det(\Phi(A))\,\det(B)\det(\Phi(B)) \\ &= \det(A\,\Phi(A))\,\det(B\,\Phi(B)) \\ &= q(A)\,q(B) \end{aligned}$$

(d) [3 marks] Prove that $f : S(C) \longrightarrow S(C)$ defined by $f(A) = p(A)\,A$ is an automorphism of $S(C)$.

Solution: first show that f is a homomorphism. Note: $p(A) = \pm 1$, which is just a scalar in terms of matrix multiplication.

$$f(AB) = p(AB)\,(AB) = p(A)\,p(B)\,A\,B = (p(A)\,A)\,(p(B)\,B) = f(A)\,f(B).$$

Since $S(C)$ is a finite group, to show f is an isomorphism we need only show it is one-to-one:

$$\begin{aligned} f(A) = I &\Rightarrow p(A)\,A = I \\ &\Rightarrow p(A)p(A)\,A = p(A)\,I \\ &\Rightarrow A = p(A)\,I, \text{ since } (p(A))^2 = 1 \\ &\Rightarrow A = I \text{ or } -I \end{aligned}$$

But $A = -I$ is an extraneous solution since

$$f(-I) = p(-I)\,(-I) = \det(\Phi(-I))\,(-I) = \det(I)\,(-I) = -I \neq I.$$

(e) [6 marks] Show that $S(C)$ has three distinct normal subgroups of order 24. Determine if any of these three subgroups are isomorphic to each other.

Solution: three normal subgroups of $S(C)$ are

$$P = \ker(p),\ Q = \ker(q),\ R = \ker(\det).$$

Since each of p, q and det is onto $\{1, -1\}$, the order of each kernel, by the First Isomorphism Theorem, is 24. P consists of the 24 matrices

$$\begin{bmatrix} \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \end{bmatrix}, \begin{bmatrix} 0 & \pm1 & 0 \\ 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \pm1 \\ \pm1 & 0 & 0 \\ 0 & \pm1 & 0 \end{bmatrix},$$

none of which has order 4. Both Q and R contain (different) elements of order 4, for example

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \in Q = \ker(q), \text{ and } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \in R = \ker(\det).$$

Thus the three groups are distinct, and P is definitely not isomorphic to Q or R. But Q and R *are* isomorphic. Use the isomorphism from part (d) and show that $f : Q \longrightarrow R$. Suppose $A \in Q$; then $\det(A\Phi(A)) = 1 \Leftrightarrow \det(A)\det(\Phi(A)) = 1$. So

$$\begin{aligned} \det(f(A)) &= \det\left(p(A)\, A\right) \\ &= (p(A))^3 \det(A) \\ &= (\det(\Phi(A)))^3 \det(A) \\ &= (\det(\Phi(A)))^2 \left(\det(\Phi(A)) \det(A)\right) \\ &= (\pm1)^2 \cdot 1 = 1 \end{aligned}$$

and $f(A) \in R$.

Aside: the matrices in P are listed above. The matrices in R are matrices that represent rotations. The matrices in Q all have an even number of -1's. Making use of these observations could furnish other ways to solve this problem. Finally, for interest:

- P is isomorphic to $A_4 \oplus \mathbb{Z}_2$
- Q and R are both isomorphic to S_4
- $S(C)$ is isomorphic to $S_4 \oplus \mathbb{Z}_2$