University of Toronto
Solutions to **MAT301H1F TERM TEST, Part 2**
Friday, October 28, 2016
Duration: 50 minutes

No aids permitted.

Instructions: Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parentheses beside the question number. **Total Marks: 25**

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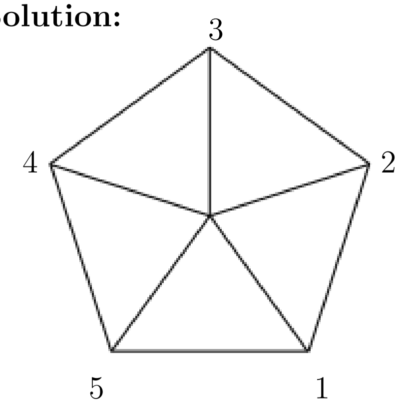
1. [5 marks] Viewing the members of D_5 as a group of permutations of a regular pentagon with consecutive vertices labeled 1, 2, 3, 4, 5, what geometric symmetry corresponds to the permutation (13524)? Which symmetry corresponds to the permutation (23)(14)?

Solution:



The cycle (13524) represents a rotation of

$$2 \times 72^\circ = 144^\circ$$

counter clockwise around the centre of the pentagon. That is,

$$1 \to 3, 2 \to 4, 3 \to 5, 4 \to 1, 5 \to 2.$$

Or you could describe it as a rotation of 216° clockwise around the centre.

The permutation (23)(14) represents a reflection in the line joining vertex 5 with the midpoint of side connecting vertices 2 and 3.

2. [10 marks] Let $\phi : G \longrightarrow H$ be a group homomorphism. Recall that

$$\ker(\phi) = \{x \in G \mid \phi(x) = e_H\} \text{ and } \mathrm{im}\,(\phi) = \{\phi(x) \mid x \in G\}.$$

(a) [4 marks] Prove that $\ker(\phi)$ is a subgroup of G.

Solution: use the subgroup test.

1. $\ker(\phi)$ is non-empty, since $\phi(e_G) = e_H \Leftrightarrow e_G \in \ker(\phi)$.
2.

$$\begin{aligned} x, y \in \ker(\phi) &\Rightarrow \phi(x) = e_H \text{ and } \phi(y) = e_H \\ &\Rightarrow \phi(x) = e_H \text{ and } \phi(y^{-1}) = (\phi(y))^{-1} = e_H^{-1} = e_H \\ &\Rightarrow \phi(xy^{-1}) = \phi(x)\phi(y^{-1}) = e_H \cdot e_H = e_H \\ &\Rightarrow xy^{-1} \in \ker(\phi) \end{aligned}$$

Thus $\ker(\phi) \leq G$.

(b) [6 marks; 2 marks for each part.] Let $\phi : \mathbb{Z}_{50} \longrightarrow \mathbb{Z}_{15}$ be the homomorphism defined by $\phi(x) = 3x$. Find the elements in each of the following:

1. $\ker(\phi)$

   Solution:

$$\begin{aligned} \ker(\phi) &= \{x \in \mathbb{Z}_{50} \mid \phi(x) \equiv 0 \bmod 15\} \\ &= \{x \in \mathbb{Z}_{50} \mid 3x \equiv 0 \bmod 15\} \\ &= \{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\} \end{aligned}$$

2. $\mathrm{im}\,(\phi)$

   Solution:

$$\mathrm{im}\,(\phi) = \{\phi(x) \in \mathbb{Z}_{15} \mid x \in \mathbb{Z}_{50}\} = \{3x \in \mathbb{Z}_{15} \mid x \in \mathbb{Z}_{50}\} = \{0, 3, 6, 9, 12\}$$

3. the left coset $7 + \ker(\phi)$

   Solution:

$$7+\ker(\phi) = 7+\{0, 5, 10, 15, 20, 25, 30, 35, 40, 45\} = \{2, 7, 12, 17, 22, 27, 32, 37, 42, 47\}$$

3. [10 marks] Recall that D_4, the dihedral group of order 8, can be described as

$$D_4 = \langle a, b \mid a^4 = b^2 = e, bab = a^3 \rangle.$$

(a) [4 marks] How many elements are there in D_4 of order 4? of order 3? of order 2?

Solution: D_4 consists of one identity; two rotations of order 4: a, a^3; one rotation of order 2: a^2; and four reflections of order 2: b, ba, ba^2, ba^3. So D_4 has

- two elements of order 4
- zero elements of order 3 (also because 3 does not divide $8 = |D_4|$)
- five elements of order 2

(b) [2 marks] Suppose $f \in \operatorname{Aut}(D_4)$ such that $f(a) = a^3$ and $f(b) = ba$. What is $f(ba^2)$?

Solution:

$$f(ba^2) = f(b)f(a)^2 = (ba)(a^3)^2 = ba \cdot a^6 = ba^7 = ba^3.$$

(c) [4 marks] How many automorphisms of D_4 are there?

Solution: if f is an automorphism, then $|f(a)| = |a|$. Thus the possibilities are

$$f(a) = a \text{ or } a^3 \text{ AND } f(b) = a^2, b, ba, ba^2 \text{ or } ba^3.$$

We must also have $f(bab) = f(a^3)$, which will be true if and only if

$$f(b)f(a)f(b) = f(a)^3 \Leftrightarrow f(b)\, a\, f(b) = a^3,$$

since $|f(b)| = 2$. The only four choices of $f(b)$ that satisfy this equation are

$$f(b) = b, ba, ba^2, ba^3.$$

For example:

$$f(b) = ba \Rightarrow f(b)af(b) = ba\, a\, ba \Rightarrow bab\, bab\, a = a^3a^3a = a^7 = a^3.$$

But $f(b) = a^2$ doesn't since

$$a^2\, a\, a^2 = a \neq a^3.$$

$$\boxed{\text{Thus there are } 2 \times 4 = 8 \text{ automorphisms of } D_4.}$$

Alternate Solution: f must take a cyclic group to a cyclic group, so

$$f(\langle a \rangle) = \langle f(a) \rangle \text{ which must be } \langle a \rangle,$$

by an exercise in the book. Taking orders of elements into account,

$$f(a) = a \text{ or } a^3 \text{ and } f(a^2) = a^2.$$

This only leaves four possibilities for $f(b)$, namely, $f(b) = b, ba, ba^2$ or ba^3.