University of Toronto
MAT301H1F TERM TEST, Part 1
Wednesday, October 25, 2017
Duration: 50 minutes

No aids permitted.

Instructions: Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parentheses beside the question number. **Total Marks: 25**

NAME: (as on your T-card) ______________________

SIGNATURE: ______________________

Solutions

CHECK YOUR TUTORIAL:

◯ TUT0101 Wed 4 PB255	◯ TUT0201 Thu 10 RW 143	◯ TUT0301 Thu 11 RW143	◯ TUT5101 Wed 5 PB255

MARKER'S REPORT:

QUESTION	MARK
Q1	
Q2	
Q3	
TOTAL	

① mark each; all or nothing.

1. [5 marks; 1 mark for each part.] Let G be a group with operation 'multiplication' and identity element e. Suppose a is an element of G. Define the following:

   (a) $Z(G)$, the center of G

   $Z(G) = \{g \text{ in } G \mid gx = xg \text{ for all } x \text{ in } G\}$

   (b) the subgroup $\langle a \rangle$ of G

   $\langle a \rangle = \{a^n \mid n \text{ in } \mathbb{Z}\}$

   or $= \{\ldots, a^{-2}, a^{-1}, 1, a, a^2, \ldots\}$

   (c) the inner automomorphism ϕ_a of G

   $\phi_a(g) = aga^{-1}$

   (d) $C(a)$, the centralizer of $a \in G$

   $C(a) = \{x \text{ in } G \mid xa = ax\}$

   (e) the order of G

   the order of G is the number of elements of the set G.

① mark for answer; ① for explanation

2. [10 marks; 2 for each part.] Find the order of the following elements in the given groups.

Ans:

(a) $7 \in \mathbb{Z}_{42}$

$2 \cdot 7 = 14$ $4 \cdot 7 = 28$
$3 \cdot 7 = 21$ $5 \cdot 7 = 35$
$6 \cdot 7 = 42 \equiv 0 \mod 42$

6

(b) $\begin{bmatrix} \cos(2\pi/7) & -\sin(2\pi/7) \\ \sin(2\pi/7) & \cos(2\pi/7) \end{bmatrix} \in GL(2, \mathbb{R})$

rotation of order 7

7

(c) $5 \in U(13)$

$5^2 = 25 \equiv 12 \equiv -1 \mod 13$
so $5^4 \equiv 1 \mod 13$

4

(d) $(137)(2546) \in S_7$

12

$|(137)| = 3$
$|(2546)| = 4$

$\text{lcm}(3, 4) = 12$

(e) $(3, 7)$ in $\mathbb{Z}_5 \bigoplus \mathbb{Z}_{14}$

10

$|3| = 5$
$|7| = 2$

$\text{lcm}(5, 2) = 10$

① for correct choice; ~~or~~ ① for valid justification.

3. [10 marks; 2 for each part.] Determine if the following statements are True or False and give a brief explanation why. Circle your choice to the right.

(a) If H and K are both subgroups of a group G, such that $|H| = 5$ and $|K| = 11$, then $|H \cap K| = 1$. **True** **False**

Let $|H \cap K| = n$,

By Lagrange's theorem: $n | 5$ and $n | 11$

$\Rightarrow n = 1$

(b) $U(14)$ is cyclic. **True** **False**

$U(14) = \{1, 3, 5, 9, 11, 13\}$

$3^2 = 9$

$3^3 = 27 \equiv 13 \equiv -1 \bmod 14$ so $|3| = 6$ $\therefore U(14) = \langle 3 \rangle$

(c) The number of elements of order 9 in $\mathbb{Z}_{72}$ is 8. **True** **False**

$\phi(9) = 9 - 3 = 6$

(d) $A_4 \approx D_6$ **True** **False**

$|A_4| = |D_6| = 12$

BUT D_6 has an order of element 6,

A_4 doesn't.

OR: D_6 has a subgroup of order 6, but A_4 doesn't (as proved in class)

(e) If every element x of a group G satisfies $x^2 = e$, then G is Abelian. **True** **False**

Let x, y in G. Then $x^2 = e$, $y^2 = e$

and $(xy)^2 = e$

$\Rightarrow xyxy = e$

$\Rightarrow yxy = x$, since $x^2 = e$

$\Rightarrow yx = xy$, since $y^2 = e$

So G is Abelian

University of Toronto
MAT301H1F TERM TEST, Part 2
Friday, October 27, 2017
Duration: 50 minutes

No aids permitted.

Instructions: Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parentheses beside the question number. **Total Marks: 25**

NAME: (as on your T-card) ____________________

SIGNATURE: ____________________

Solutions

CHECK YOUR TUTORIAL:

◯ TUT0101 Wed 4 PB255	◯ TUT0201 Thu 10 RW 143	◯ TUT0301 Thu 11 RW143	◯ TUT5101 Wed 5 PB255

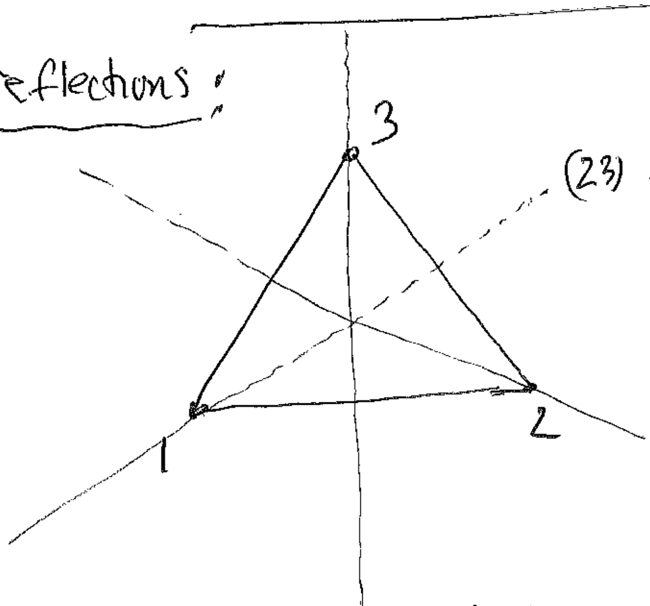
MARKER'S REPORT:

QUESTION	MARK
Q1	
Q2	
Q3	
TOTAL	

1. [10 marks] Recall that in class we proved that $S_3 \approx D_3$, where S_3 is the symmetric group of degree 3 and D_3 is the dihedral group of order 6. Write down all six permutations in S_3 and then interpret each one as a symmetry of an equilateral triangle with vertices labeled 1, 2 and 3.

$$S_3 = \{ (1), (12), (13), (23), (123), (132) \}$$ 4 marks

reflections:



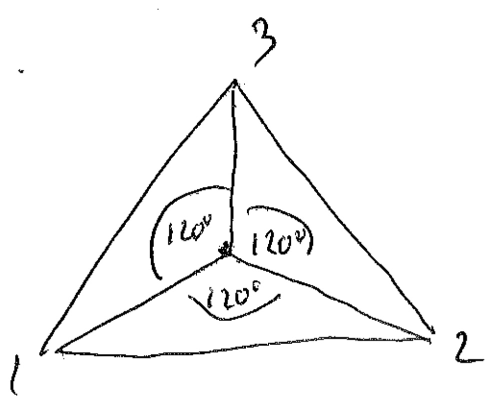
(23) is reflection in this axis

3 marks

(13) is reflection in this axis

(12) is reflection in this axis

rotations:



(1) is the identity, or 0° rotation

(123) is a counterclockwise rotation of 120° around centre of triangle

(132) is a clockwise rotation of 120° around centre of triangle

3 marks

2. [4 marks] State Lagrange's Theorem.

2 { Let G be a group with finite order;
let H be a subgroup of G

Then 1) the order of H divides the order of G
1 (ie. $|H| \,/\, |G|$)

1 2) the number of distinct left (right) cosets of H is $|G|/|H|$.

3. [11 marks] Making use of Lagrange's Theorem (or otherwise) prove the following:

(a) [3 marks] If G is a group with finite order and a is an element of G, then the order of a divides the order of G.

Let $H = \langle a \rangle$; H is a subgroup of G and $|H| = |a|$. 2

By Lagrange's theorem, $|H| \,/\, |G|$ 1
$\Leftrightarrow |a| \,/\, |G|$.

$\boxed{\text{short way}}$: if x in G, $x \neq e$, and $x^{-1} \neq e$, then $x \neq x^{-1}$.

So $G = \{1, x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_k, x_k^{-1}\}$

$\Rightarrow$ $|G| = 2k+1$, which is odd.

(b) [4 marks] If a group G has order 12 then it must have an element of order 2.

Proof: by contradiction.

Suppose for all x in G, $x \neq e$, $|x| \neq 2$. Then, by part (a), $|x| = 3, 4, 6$ or 12. BUT: $|x| = 4 \Rightarrow |x^2| = 2$

$|x| = 6 \Rightarrow |x^3| = 2$

$|x| = 12 \Rightarrow |x^6| = 2$

(2) So: $|x| = 3$, which means G has 11 elements of order 3.

This contradicts a Theorem in the book which says number of elements of order 3 is $k\phi(3) = 2k$, even

(2) for getting contradiction

OR each $x \neq e \Rightarrow \langle x \rangle = \{1, x, x^2\} \leq G$, since each subgroup of order 3 contains 2 elements of order 3, so the number of such subgroups must be $\frac{11}{2} = 5.5$; contradiction.

(c) [4 marks] If $f : S_3 \longrightarrow D_4$ is a group homomorphism, then $|\ker(f)| = 3$ or 6, where D_4 is the dihedral group of order 8.

Recall: if $f : G \to H$ is a homomorphism then $|f(x)| \,\big|\, |x|$ ie. order of $f(x)$ divides order of x

~~Observe~~ So $|f((123))| \,\big|\, |(123)| = 3$

$\Rightarrow |f((123))| = 1$ or 3

But 3 does not divide $|D_4| = 8$,

so $|f((123))| = 1$ and (123) in $\ker(f)$

Similarly, (132) in $\ker(f)$

Thus $\{1, (123), (132)\} \subset \ker(f)$ (2)

$\Rightarrow |\ker(f)| \geq 3$

By Lagrange, $|\ker(f)| = 3$ or 6. (2)