MAT188H1F - Linear Algebra - Fall 2019

Solutions to Term Test 1 - October 1, 2019

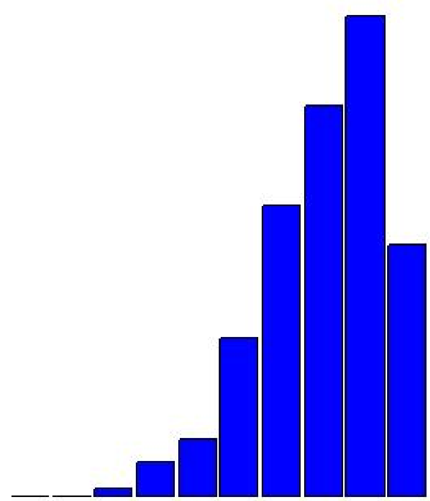
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

- Generally speaking, the results on this test were very good.
- The only question that did not have a passing average was Question 7: three short proofs.
-
-

Breakdown of Results: 931 registered students wrote this test. The marks ranged from 27.5% to 100%, and the average was 59.76 out of 80, or 74.7%. There were two perfect papers. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	15.0%
A	43.8%	80-89%	28.8%
B	23.4%	70-79%	23.4%
C	17.4%	60-69%	17.4%
D	9.5%	50-59%	9.5%
F	5.8%	40-49%	3.4%
		30-39%	2.0%
		20-29%	0.4%
		10-19%	0.0%
		0-9%	0.0%



1. [10 marks; each part is worth 2 marks. Avg: 8.5/10] Calculate the following:

(a) $\left\| \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix} \right\| = \sqrt{2^2 + 6^2 + 3^2} = \sqrt{49} = 7.$

(b) the rank of the matrix: $\begin{bmatrix} 3 & 5 & 1 & -1 \\ -1 & 0 & 2 & 3 \\ 1 & 5 & 5 & 5 \\ 0 & 3 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 0 & 5 & 7 & 8 \\ -1 & 0 & 2 & 3 \\ 0 & 5 & 7 & 8 \\ 0 & 3 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 7 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$

There are three leading entries, so the rank of the matrix is 3.

(c) $\text{proj}_{\vec{d}}(\vec{v})$, if $\vec{d} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix}$.

Soluton: $\text{proj}_{\vec{d}}(\vec{v}) = \dfrac{\vec{v} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \left(\dfrac{6 - 6 + 6}{3} \right) \vec{d} = 2\,\vec{d} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 7 \\ -3 \end{bmatrix} \times \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 - 6 \\ -(1 + 15) \\ -2 - 35 \end{bmatrix} = \begin{bmatrix} 1 \\ -16 \\ -37 \end{bmatrix}$

(e) a vector equation of the line passing through the two points $P(4, 1, 1)$ and $Q(5, 6, 7)$.

Solution: for parameter t,

$$\vec{x} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + t\overrightarrow{PQ} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix} \text{ or } \vec{x} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}.$$

There are many other possible correct answers.

2. [avg: 8.8/10] Consider the problem:

   A child has a jar full of coins, consisting of nickels, dimes and quarters. Altogether there are 180 coins in the jar. The number of dimes is one half of the total number of nickels and quarters. If the total value of the coins in the jar is \$16.00, how many of each type of coin does the child have?

   (a) [4 marks] Introduce three variables and write down a system of three linear equations in three variables that represents this problem.

   Solution: let x be the number of nickels, y the number of dimes, and z the number of quarters in the jar. Then

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 180 \\ x & - & 2y & + & z & = & 0 \\ 5x & + & 10y & + & 25z & = & 1600 \end{array}$$

   Of course, the equations could be in any order.

   (b) [6 marks] Solve the problem by finding the reduced row echelon form of the augmented matrix of the system of equations you wrote down in part (a).

   Solution: reduce the augmented matrix to reduced row echelon form:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 180 \\ 1 & -2 & 1 & 0 \\ 5 & 10 & 25 & 1600 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 180 \\ 0 & -3 & 0 & -180 \\ 0 & 5 & 20 & 700 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 180 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 20 & 400 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 100 \\ 0 & 1 & 0 & 60 \\ 0 & 0 & 1 & 20 \end{array}\right].$$

   Thus $x = 100, y = 60$ and $z = 20$. That is, the child has

   - 100 nickels
   - 60 dimes
   - 20 quarters

3. [avg: 6.9/10]

3.(a) [4 marks] Find all values of a so that the vector $\vec{u} = \begin{bmatrix} -1/9 \\ -4/9 \\ a \end{bmatrix}$ is a unit vector.

Solution:

$$\|\vec{u}\|^2 = 1 \Rightarrow \frac{1}{81} + \frac{16}{81} + a^2 = 1 \Rightarrow a^2 = \frac{64}{81} \Rightarrow a = \pm\frac{8}{9}.$$

3.(b) [6 marks] Consider a system of homogeneous linear equations with exactly two basic solutions,

$$\vec{u} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Find all values of b so that $\vec{w} = \begin{bmatrix} 2 \\ b \\ 5 \end{bmatrix}$ is also a solution to the same system of homogeneous linear equations.

Solution: $\vec{w}$ must be a linear combination of $\vec{u}$ and $\vec{v}$. Let $\vec{w} = s\,\vec{u} + t\,\vec{v}$. Thus

$$\begin{bmatrix} 2 \\ b \\ 5 \end{bmatrix} = s \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3s + t \\ 2s - t \\ t \end{bmatrix}.$$

Comparing first and third components we have $2 = 3s + t$ and $t = 5$. Thus $3s = -3 \Leftrightarrow s = -1$. Then

$$b = 2(-3) - 1 = -7$$

is the only possible value for b.

Alternate Solution: since we are in $\mathbb{R}^3$, the set of solutions to a homogeneous system with two basic solutions is a plane with direction vectors $\vec{u}$ and $\vec{v}$. Then the normal to the plane is

$$\vec{n} = \vec{u} \times \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -5 \end{bmatrix}$$

and the scalar equation of the plane is $2x - 3y - 5z = 0$. Then $\vec{w}$ is in the plane if

$$2(2) - 3(b) - 5(5) = 0 \Leftrightarrow -21 = 3b \Leftrightarrow b = -7,$$

as before.

4. [10 marks; avg: 8.0/10] Show that the solution set to the system of three equations in three variables

$$\begin{cases} 2x_1 & + & x_2 & - & x_3 & = & 9 \\ 3x_1 & - & x_2 & - & 2x_3 & = & 3 \\ & & 5x_2 & + & x_3 & = & 21 \end{cases}$$

represents a line in $\mathbb{R}^3$. What is a direction vector of the line?

Solution: solve the system any way you like. Here is one approach:

1. First note that three times the first equation minus two times the second gives you the third equation:

$$6x_1 - 6x_1 + 3x_2 - (-2x_2) - 3x_3 - 2(-2x_3) = 27 - 6 \Leftrightarrow 5x_2 + x_3 = 21.$$

   Thus the third equation is redundant and the system of equations will have infinitely many solutions.

2. Adding the first two equations gives you

$$5x_1 - 3x_3 = 12.$$

3. Let $x_3 = t$ be a parameter. Using the previous equation, and the given third equation we have

$$5x_1 = 12 + 3t \text{ and } 5x_2 = 21 - t.$$

Hence the solution to the system of equations is given by

$$\begin{cases} x_1 & = & \frac{12}{5} & + & \frac{3}{5}t \\ x_2 & = & \frac{21}{5} & - & \frac{1}{5}t \\ x_3 & = & & & t \end{cases},$$

which are parametric equations of a line in $\mathbb{R}^3$. You must state this! For a direction vector take

$$\begin{bmatrix} 3/5 \\ -1/5 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 3 \\ -1 \\ 5 \end{bmatrix}.$$

Alternate Solution: reduce the augmented matrix of the system to reduced row echelon form:

$$\left[\begin{array}{rrr|r} 2 & 1 & -1 & 9 \\ 3 & -1 & -2 & 3 \\ 0 & 5 & 1 & 21 \end{array}\right] \to \left[\begin{array}{rrr|r} 2 & 1 & -1 & 9 \\ 1 & -2 & -1 & -6 \\ 0 & 5 & 1 & 21 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -2 & -1 & -6 \\ 0 & 5 & 1 & 21 \\ 0 & 5 & 1 & 21 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -3/5 & 12/5 \\ 0 & 1 & 1/5 & 21/5 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Then proceed as before: let $x_3 = t$ be a parameter, and solve for x_1 and x_2 in terms of t.

5. [avg: 8.9/10] Consider the three points $P(2,-4,5), Q(6,-3,8), R(6,-2,9)$ with coordinates as indicated.

(a) [6 marks] Find the scalar equation of the plane that contains the three points.

Solution: to get a normal vector $\vec{n}$ to the plane take the cross product of any pair of vectors in the plane, namely any pair of $\overrightarrow{PQ}, \overrightarrow{PR}$ or $\overrightarrow{RQ}$. For instance, take

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 4 \end{bmatrix}.$$

Picking P as a point on the plane, the scalar equation of the plane is

$$-2x - 4y + 4x = (-2)(2) - 4(-4) + 4(5) \Leftrightarrow x + 2y - 2z = -16.$$

Alternate Approach: the vector equation of the plane is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix} + s \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 2+4s+4t \\ -4+s+2t \\ 5+3s+4t \end{bmatrix}.$$

Eliminate parameters: $3y - z = -17 + 2t$ and $x - 4y = 18 - 4t$; then

$$6y - 2z + x - 4y = -34 + 18 \Leftrightarrow x + 2y - 2z = -16,$$

as before.

Another Approach: let the equation of the plane be $ax + by + cz = d$, substitute all three given points, and solve for a, b, c and d. (Not recommended.)

(b) [4 marks] What is the area of the triangle with vertices P, Q and R?

Solution: the area of the triangle ΔPQR is half of the area of the parallelogram determined by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$. Thus the area of the triangle is given by

$$A = \frac{1}{2}\|\overrightarrow{PQ} \times \overrightarrow{PR}\| = \frac{1}{2}\left\| \begin{bmatrix} -2 \\ -4 \\ 4 \end{bmatrix} \right\| = \frac{2}{2}\left\| \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\| = \sqrt{9} = 3,$$

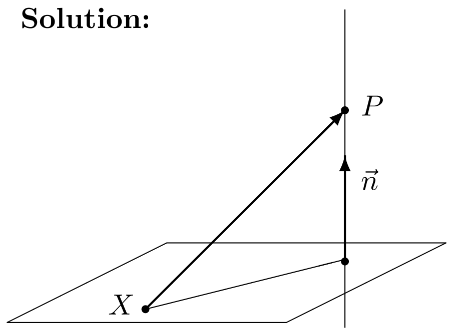
where we have used the calculation of $\overrightarrow{PQ} \times \overrightarrow{PR}$ from part (a).

6. [avg: 8.3/10]

6.(a) [4 marks] Find the minimum distance from the point $P(3,1,-1)$ to the plane with equation

$$2x + y - z = 6.$$

Solution:



- $\vec{n} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ is a normal vector to the plane.
- $X(3,0,0)$ is a point on the plane.
- The minimum distance from P to the plane is given by $D = \|\text{proj}_{\vec{n}}\left(\overrightarrow{XP}\right)\|$. Calculating gives

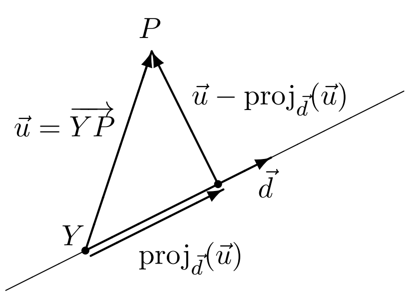
$$\overrightarrow{XP} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}; D = \left\| \frac{\overrightarrow{XP} \cdot \vec{n}}{\|\vec{n}\|^2} \vec{n} \right\| = \frac{|\overrightarrow{XP} \cdot \vec{n}|}{\|\vec{n}\|} = \frac{2}{\sqrt{6}}.$$

6.(b) [6 marks] Find the minimum distance from the point $P(3,1,-1)$ to the line with vector equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix},$$

where t is a parameter.

Solution:



- $\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ is a direction vector of the line.
- $Y(3,3,5)$ is a point on the line.
- Let $\vec{u} = \overrightarrow{YP}$.
- Then the minimum distance from the point P to the line is given by $D = \|\vec{u} - \text{proj}_{\vec{d}}(\vec{u})\|$. Calculating gives

$$\vec{u} = \begin{bmatrix} 0 \\ -2 \\ -6 \end{bmatrix}; D = \|\vec{u} - \text{proj}_{\vec{d}}(\vec{u})\| = \left\| \begin{bmatrix} 0 \\ -2 \\ -6 \end{bmatrix} - \frac{2-18}{4+1+9} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\| = \frac{2}{7} \left\| \begin{bmatrix} 8 \\ -11 \\ -9 \end{bmatrix} \right\| = \frac{2}{7}\sqrt{266}.$$

7. [10 marks; avg: 4.5/10] For this problem assume all vectors are in $\mathbb{R}^3$. Prove the following:

(a) [3 marks] If $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors then the volume of the parallelepiped with sides $\vec{u}, \vec{v}$ and $\vec{u} \times \vec{v}$ is 1.

Proof: the volume of the parallelepiped with sides $\vec{u}, \vec{v}, \vec{w}$ is given by the absolute value of their triple product: $V = |\vec{w} \cdot \vec{u} \times \vec{v}|$. Letting $\vec{w} = \vec{u} \times \vec{v}$, we get

$$V = |\vec{u} \times \vec{v} \cdot \vec{u} \times \vec{v}| = \|\vec{u} \times \vec{v}\|^2 = (\|\vec{u}\|\|\vec{v}\| \sin(\pi/2))^2 = 1,$$

since it is given that $\vec{u}$ and $\vec{v}$ are orthogonal unit vectors.

Alternate Approach: if you take $V = |\vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v}))|$. Then the calculations are

$$\begin{aligned} V &= \|\vec{u}\|\|\vec{v} \times (\vec{u} \times \vec{v})\|, \text{ but need to explain why } \vec{u} \text{ and } \vec{v} \times (\vec{u} \times \vec{v}) \text{ are parallel} \\ &= \|\vec{u}\|\|\vec{v}\|\|\vec{u} \times \vec{v}\| \sin(\pi/2), \text{ because } \vec{v} \text{ and } \vec{u} \times \vec{v} \text{ are orthogoanl} \\ &= \|\vec{u}\|\|\vec{v}\|\|\vec{u}\|\|\vec{v}\|, \text{ since } \vec{u} \text{ and } \vec{v} \text{ are orthogonal} \\ &= 1, \text{ since } \vec{u}, \vec{v} \text{ are both unit vectors} \end{aligned}$$

(b) [3 marks] If $\vec{d}$ is a non-zero vector then $\mathrm{proj}_{\vec{d}}\big(\mathrm{proj}_{\vec{d}}(\vec{v})\big) = \mathrm{proj}_{\vec{d}}(\vec{v})$.

Proof: $\mathrm{proj}_{\vec{d}}(\vec{v}) = \dfrac{\vec{v} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\, \vec{d}$, so

$$\mathrm{proj}_{\vec{d}}\big(\mathrm{proj}_{\vec{d}}(\vec{v})\big) = \frac{\frac{\vec{v} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\, \vec{d} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\, \vec{d} = \frac{\vec{v} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\, \vec{d} = \mathrm{proj}_{\vec{d}}(\vec{v}).$$

(c) [4 marks] If $\|\vec{u}\| = 7, \|\vec{v}\| = 3$ and $\vec{u} \cdot \vec{v} = -12.75$, then $\|3\vec{u} + 2\vec{v}\| = 18$.

Proof:

$$\begin{aligned} \|3\vec{u} + 2\vec{v}\|^2 &= (3\vec{u} + 2\vec{v}) \cdot (3\vec{u} + 2\vec{v}) \\ &= 9\, \vec{u} \cdot \vec{u} + 12\, \vec{u} \cdot \vec{v} + 4\, \vec{v} \cdot \vec{v} \\ &= 9\, \|\vec{u}\|^2 + 12\, \vec{u} \cdot \vec{v} + 4\, \|\vec{v}\|^2 \\ &= 9(49) + 12(-12.75) + 4(9) \\ &= 324 \\ \Rightarrow \|3\vec{u} + 2\vec{v}\| &= \sqrt{324} = 18 \end{aligned}$$

Alternate Solution: use cosine law. Let the angle between $\vec{u}$ and $\vec{v}$ be θ. Then

$$\|3\vec{u} + 2\vec{v}\|^2 = \|3\vec{u}\|^2 + \|\vec{2v}\|^2 - 2\|3\vec{u}\|\|2\vec{v}\| \cos(\pi - \theta) = 9\|\vec{u}\|^2 + 4\|\vec{v}\|^2 + 12\ \vec{u} \cdot \vec{v} = 324$$

8. [avg: 5.8/10] Indicate if the following statements are **True** or **False**, and give a *brief* explanation why.

(a) [2 marks] Non-zero vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^3$ are parallel if $|\vec{u} \cdot \vec{v}| = \|\vec{u}\|\|\vec{v}\|$. $\otimes$ **True** $\bigcirc$ **False**

Justification: $|\vec{u} \cdot \vec{v}| = \|\vec{u}\|\|\vec{d}\|\,|\cos\theta| = \|\vec{u}\|\|\vec{v}\| \Leftrightarrow |\cos\theta| = 1 \Leftrightarrow \theta = 0$ or π. That is, the angle between the vectors is $\theta = 0$, in which case they are parallel, or $\theta = \pi$, in which case they are in the opposite direction but still parallel.

(b) [2 marks] The points $P(1, 8, 8), Q(7, -6, 10), R(4, 1, 9)$ are on the same line. $\otimes$ **True** $\bigcirc$ **False**

Justification: $\overrightarrow{PQ} = \begin{bmatrix} 6 \\ -14 \\ 2 \end{bmatrix}; \overrightarrow{PR} = \begin{bmatrix} 3 \\ -7 \\ 1 \end{bmatrix}$; therefore $\overrightarrow{PQ} = 2\overrightarrow{PR}$.

(c) [2 marks] The line with parametric equations $x = 2 + t, y = -t, z = 1 + 7t$ is orthogonal to the plane with scalar equation $3x - 4y - z = 8$. $\bigcirc$ **True** $\otimes$ **False**

Justification: a direction vector of the line is $\vec{d} = \begin{bmatrix} 1 \\ -1 \\ 7 \end{bmatrix}$, a normal vector to the line is $\vec{n} = \begin{bmatrix} 3 \\ -4 \\ -1 \end{bmatrix}$, and $\vec{d} \cdot \vec{n} = 0$. Thus the line is actually **parallel** to the plane.

(d) [2 marks] If the augmented matrix of a system of 4 homogeneous linear equations in 5 variables has rank 3, then two of the variables must be free variables. $\otimes$ **True** $\bigcirc$ **False**

Justification: a homogeneous system is always consistent, so if the rank of its augmented matrix is 3, then there are 3 leading variables, and hence $5 - 3 = 2$ free variables.

(e) [2 marks] If the augmented matrix of a system of 4 linear equations in 5 variables has rank 3, then the system of equations must have infinitely many solutions. $\bigcirc$ **True** $\otimes$ **False**

Justification: the system may not be consistent. For example the matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

has rank 3 and is the augmented matrix of a system of 4 equations in 5 variables that has no solutions.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.