MAT188H1F - Linear Algebra - Fall 2018

Solutions to Term Test 1 - October 2, 2018

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Genreal Comments:

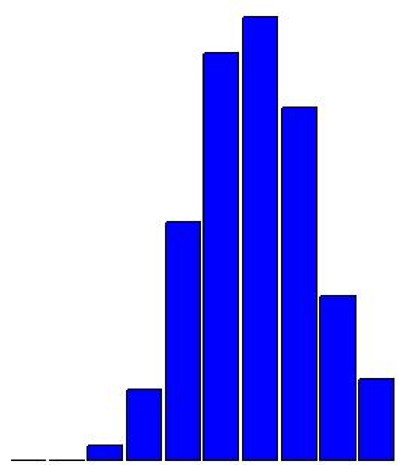
- Questions 1, 2, 3 and 4 were well done on the whole. The median of all four questions was at least 9.
- The averages on Questions 5 and 8 were close to 5 out of 10, typical for these kinds of questions.
- Questions 6 and 7 both had an average close to 3; 372 students had 0 on Question 6, 334 had 0 on Question 7. Question 7 involved proving so these results are not a surprise, but Question 6 was basically computational, so the low results are a big surprise.
- In Question 3, many students wrote down the augmented matrix of the system

$$\begin{aligned} ax_1 \quad + (2b+1)x_2 &= \ \ 1 \\ (a+b)x_1 - \quad 4x_2 \quad &= -8 \end{aligned}.$$

  and proceeded to reduce it. This is irrelevant and a waste of time! Those who did this, invariably divided by expressions in terms of a and b which could be zero, but did not take this into account.

Breakdown of Results: 830 registered students wrote this test. The marks ranged from 21.25% to 100%, and the average was 63%. There were three perfect papers. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.6%
A	13.9%	80-89%	9.3%
B	20.0%	70-79%	20.0%
C	25.0%	60-69%	25.0%
D	22.9%	50-59%	22.9%
F	18.2%	40-49%	13.4%
		30-39%	4.0%
		20-29%	0.8%
		10-19%	0.0%
		0-9%	0.0%



1. [10 marks; avg: 8.18/10] Let

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ -2 & 1 & 3 \end{bmatrix}.$$

Determine whether each of the following expressions is defined. If so, simplify the expression into a single matrix. If not, explain why it is not defined.

(a) $(A+B)C$

Solution: not defined. A and B have different sizes, so they cannot be added.

(b) $AC - (3B)^T$

Solution:

$$\begin{aligned} AC - (3B)^T &= \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 2 \\ -2 & 1 & 3 \end{bmatrix} - \left(3 \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} \right)^T \\ &= \begin{bmatrix} -2 & 1 & 9 \\ -5 & 2 & 11 \end{bmatrix} - \begin{bmatrix} 3 & 3 & 6 \\ 3 & 0 & 9 \end{bmatrix} = \begin{bmatrix} -5 & -2 & 3 \\ -8 & 2 & 2 \end{bmatrix} \end{aligned}$$

(c) $ACB + (B^T)B$

Solution: you can use your calculation of AC from part (b).

$$\begin{aligned} ACB + (B^T)B &= \begin{bmatrix} -2 & 1 & 9 \\ -5 & 2 & 11 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 17 & 25 \\ 19 & 28 \end{bmatrix} + \begin{bmatrix} 6 & 7 \\ 7 & 10 \end{bmatrix} = \begin{bmatrix} 23 & 32 \\ 26 & 38 \end{bmatrix} \end{aligned}$$

(d) $AB - BA$

Solution: not defined. AB is 2×2, BA is 3×3, so AB and BA cannot be subtracted.

2. [10 marks; avg: 9.45/10] Consider the problem

A total of 275 people attend a concert. Ticket prices are \$12 for adults, \$10 for seniors and \$8 for students. The total revenue was \$3100. Determine how many adults, seniors and students attended the concert, given that the number of seniors who attended was twice the number of students.

(a) [4 marks] Introduce three variables and write down a system of three linear equations in three variables that represents this problem.

Solution: let x be the number of adults who attended, y the number of seniors, z the number of students. Then

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 275 \\ 12x & + & 10y & + & 8z & = & 3100 \\ & & y & - & 2z & = & 0 \end{array}$$

(b) [6 marks] Solve the problem by finding the reduced row echelon form of the augmented matrix of the system of equations you wrote down in part (a).

Solution:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & 275 \\ 12 & 10 & 8 & 3100 \\ 0 & 1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 275 \\ 0 & 2 & 4 & 200 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 1 & 1 & 275 \\ 0 & 1 & 2 & 100 \\ 0 & 0 & 4 & 100 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 1 & 0 & 250 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 25 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 50 \\ 0 & 0 & 1 & 25 \end{array}\right], \text{ which is in RREF}$$

Therefore, $x = 200, y = 50, z = 25$; that is

- 200 adults attended the concert,
- 50 seniors attended the concert,
- 25 students attended the concert.

3. [10 marks; avg: 8.53/10]

3.(a) [4 marks] Find all values of c so that the system of linear equations

$$\begin{aligned} x_2 + x_3 &= 2 \\ x_1 + x_2 + 2x_3 &= 5 \\ x_1 - x_2 \qquad &= 1 + c \end{aligned}$$

has a solution.

Solution: reduce the augmented matrix, checking for consistency:

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 5 \\ 1 & -1 & 0 & 1+c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & -1 & 0 & 1+c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4-c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & c \end{array}\right].$$

The system will have a solution if and only if $c = 0$.

Alternate Solution: observe that on the left side of equal signs in the system of equations, the third equation is the second minus two times the first; so on the right side of the equal signs, for consistency, $1 + c = 5 - 2(2) = 1 \Leftrightarrow c = 0$.

3.(b) [6 marks] Find a and b such that $x_1 = 3, x_2 = 2$ is a solution of the system

$$\begin{aligned} ax_1 \quad + (2b+1)x_2 &= 1 \\ (a+b)x_1 - \quad 4x_2 \quad &= -8 \end{aligned}.$$

Solution: substitute $x_1 = 3, x_2 = 2$ into the given equations and solve the new system of equations for a and b :

$$\left.\begin{array}{ccccc} 3a & + & 2(2b+1) & = & 1 \\ 3(a+b) & - & 8 & = & -8 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{ccccc} 3a & + & 4b & = & -1 \\ 3a & + & 3b & = & 0 \end{array}\right.$$

Subtracting the last two equations gives $b = -1$; and consequently $a = 1$, since $a = -b$.

4. [10 marks; avg: 8.51/10]

4.(a) [3 marks] Write the vector $\vec{u} = \begin{bmatrix} 13 \\ 8 \\ -3 \end{bmatrix}$ as a linear combination of $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution: let $\vec{u} = s\vec{v}_1 + t\vec{v}_2$. Reduce the augmented matrix of the system of equations with this vector equation:

$$\left[\begin{array}{cc|c} 2 & 1 & 13 \\ 1 & 1 & 8 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 13 \\ 0 & -1 & -3 \\ 0 & -1 & -3 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 0 & 10 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

So $s = 5$ and $t = 3$, and $\vec{u} = 5\vec{v}_1 + 3\vec{v}_2$. (Note: this question is easy enough to be done by 'inspection.')

4.(b) [7 marks] Solve the homogeneous system of equations

$$\begin{cases} x_1 + 2x_2 + 5x_3 + 6x_4 = 0 \\ 2x_1 + x_2 + 4x_3 + 9x_4 = 0 \\ 2x_1 + 7x_2 + 16x_3 + 15x_4 = 0 \end{cases}$$

by reducing its augmented matrix to reduced row echelon form and then expressing the solution as a linear combination of basic solutions.

Solution:

$$\left[\begin{array}{cccc|c} 1 & 2 & 5 & 6 & 0 \\ 2 & 1 & 4 & 9 & 0 \\ 2 & 7 & 16 & 15 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 5 & 6 & 0 \\ 0 & 3 & 6 & 3 & 0 \\ 0 & -3 & -6 & -3 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 2 & 5 & 6 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 4 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \text{ which is in RREF}$$

Let $x_3 = s, x_4 = t$ be parameters. Then the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -s - 4t \\ -2s - t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

5. [10 marks; avg: 5.4/10] Consider the system of equations

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ x_1 + (a+2)x_2 + 2x_3 = 5 \\ x_1 + 7x_2 + (a+1)x_3 = 9 \end{cases},$$

where a is in $\mathbb{R}$. Find all value(s) of a for which the system of equations has

(a) no solution, (b) a unique solution, (c) infinitely many solutions.

Solution: reduce the augmented matrix of the system and analyze how many leading entries there can be.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & a+2 & 2 & 5 \\ 1 & 7 & a+1 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & a+1 & 1 & 4 \\ 0 & 6 & a & 8 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 6 & a & 8 \\ 0 & a+1 & 1 & 4 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 6 & a & 8 \\ 0 & 0 & 6-a(a+1) & 24-8(a+1) \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 6 & a & 8 \\ 0 & 0 & a^2+a-6 & 8a-16 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 6 & a & 8 \\ 0 & 0 & (a+3)(a-2) & 8(a-2) \end{array}\right]$$

(a) If $a = -3$, then the last row of the reduced matrix is $[0\ 0\ 0 \mid -40]$, which indicates that the system has no solution.

(b) If $a \neq -3, a \neq 2$ then the reduced matrix has rank 3 and each variable is a leading variable, so the system has a unique solution.

(c) If $a = 2$ then the last row is a row of all zeros, the reduced matrix has rank 2 and there is one free variable, so the system has infinitely many solutions.

6. [10 marks; avg: 3.05/10] Let

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 3 & 0 \\ -1 & 4 \end{bmatrix};$$

let $T_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ and $T_B : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the matrix transformations induced by A and B, respectively.

(a) [5 marks] Show that if $\vec{x} \in \mathbb{R}^2$ and $(T_A \circ T_B)(\vec{x}) = \vec{0}$, then $\vec{x} = \vec{0}$.

Solution: $(T_A \circ T_B)(\vec{x}) = AB\,\vec{x}$; solve the homogeneous system with augmented matrix $[AB|\,\vec{0}]$.

$$AB = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 12 \\ 10 & 8 \end{bmatrix},$$

and

$$\left[\begin{array}{cc|c} -3 & 12 & 0 \\ 10 & 8 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} -1 & 4 & 0 \\ 0 & 48 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

So indeed, $\vec{x} = \vec{0}$.

(b) [5 marks] Suppose $\vec{y}$ is a vector in $\mathbb{R}^3$ such that $(T_B \circ T_A)(\vec{y}) = \vec{0}$. Must $\vec{y} = \vec{0}$? Justify your answer.

Solution: $(T_B \circ T_A)(\vec{y}) = BA\,\vec{y}$; solve the homogeneous system with augmented matrix $[BA|\,\vec{0}]$.

$$BA = \begin{bmatrix} 2 & 4 \\ 3 & 0 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 2 & 0 \\ 3 & -3 & 6 \\ 11 & 5 & -6 \end{bmatrix},$$

and

$$\left[\begin{array}{ccc|c} 14 & 2 & 0 & 0 \\ 3 & -3 & 6 & 0 \\ 11 & 5 & -6 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 16 & -28 & 0 \\ 0 & 16 & -28 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 4 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This last matrix indicates that the system of homogeneous equations has one free variable, so there are infinitely many solutions $\vec{y}$ such that $(T_B \circ T_A)(\vec{y}) = \vec{0}$. That is, $\vec{y}$ need not be $\vec{0}$. For example

$$\vec{y} = \begin{bmatrix} -1 \\ 7 \\ 4 \end{bmatrix}$$

is a non-zero solution to $(T_B \circ T_A)(\vec{y}) = \vec{0}$.

7. [10 marks; avg: 2.82/10]

7.(a) [4 marks] Let A and B be two $n \times n$ symmetric[1] matrices. Prove that AB is symmetric if and only if $AB = BA$.

Solution: we are given $A^T = A$ and $B^T = B$. Then

$$\begin{aligned} (AB)^T = AB \quad &\Leftrightarrow \quad B^T A^T = AB \\ &\Leftrightarrow \quad BA = AB \end{aligned}$$

7.(b) [6 marks] Let A be a 2×2 matrix. Prove that

$$AA^T = A^T A \text{ if and only if } A \text{ is symmetric or } A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix},$$

for some real numbers a and b.

Solution: let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; then $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. So

$$\begin{aligned} AA^T = A^T A \quad &\Leftrightarrow \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &\Leftrightarrow \quad \begin{bmatrix} a^2 + b^2 & ac + bd \\ ca + db & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ba + dc & b^2 + d^2 \end{bmatrix} \\ &\Leftrightarrow \quad c^2 = b^2 \text{ and } ac + bd = ab + cd \\ &\Leftrightarrow \quad c = b, \text{ or } c = -b \text{ and } bd = ab \end{aligned}$$

Consider the three cases:

- if $c = b$ then $A = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$, which is symmetric;
- if $c = -b$ and $b = 0$, then $A = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, which is also symmetric;
- if $c = -b$ and $b \neq 0$, then $d = a$, and $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

[1]Recall: an $n \times n$ matrix M is symmetric if $M^T = M$.

8. [10 marks; avg: 4.47/10] Indicate if the following statements are **True** or **False**, and give a *brief* explanation why. If the statement is **True** you must give a short proof; if the statement is **False** you should give a counterexample.

(a) [2 marks] If the zero vector is a solution to a system of linear equations then the system must be homogeneous. ⊗ **True** ○ **False**

Explanation: let the system be $A\vec{x} = \vec{b}$. If $\vec{x} = \vec{0}$ is a solution, then $\vec{b} = A\vec{0} = \vec{0}$.

(b) [2 marks] If $\vec{x}$ is a non-zero vector and both $\vec{x}$ and $2\,\vec{x}$ are solutions to the same system of linear equations, then the system must be homogeneous. ⊗ **True** ○ **False**

Explanation: let the system be $A\vec{x} = \vec{b}$. If $\vec{x}$ and $2\vec{x}$ are both solutions then

$$\vec{b} = A(2\vec{x}) = 2A\vec{x} = 2\vec{b} \Rightarrow \vec{b} = 2\vec{b} \Rightarrow \vec{b} = 0.$$

(c) [2 marks] If A is the augmented matrix of a system of linear equations and the rank of A is equal to the number of equations in the system, then the system of equations is consistent.

○ **True** ⊗ **False**

Counter example: $\begin{cases} x + y + z = 1 \\ x + y + z = 2 \end{cases}$ has no solution,

but the rank of its augmented matrix in reduced row echelon form, $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$, is 2, which is the number of equations in the system.

(d) [2 marks] If the rank of the constant vector $\vec{b}$ of a system of linear equations $A\,\vec{x} = \vec{b}$ is zero, then the system is consistent. ⊗ **True** ○ **False**

Explanation: if $\vec{b}$, as a matrix, has rank zero, then $\vec{b} = \vec{0}$. So the system is homogeneous and has (at least) the trivial solution.

(e) [2 marks] If a system of 3 linear equations in 4 variables is consistent, then it must have infinitely many solutions. ⊗ **True** ○ **False**

Explanation: the rank of the augmented matrix of the system is at most 3, so the system has at least one free variable; so it has infinitely many solutions.