MAT188H1F - Linear Algebra - Fall 2015

Solutions to Term Test 1 - October 6, 2015

Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

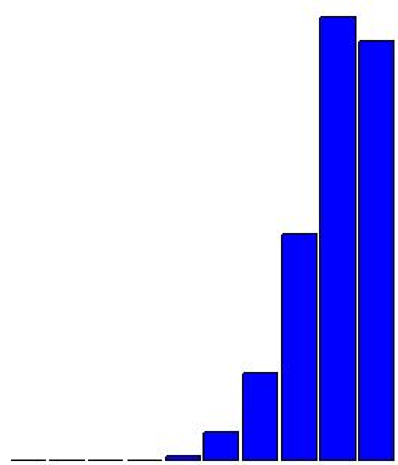
1. The results on this test were very good. An average of 83.8% is just like old times! The challenge is to maintain these good results on the next test.

2. In the True or False questions, Question 8, you should know by now that you can't justify the truth of a statement by giving an example. You must give a general argument.

3. Many of you still don't realize that it is not enough to plop down symbols, expressions, or equations. You must LEAD the reader through your solution; that's what "Present complete solutions" means.

Breakdown of Results: 933 students wrote this test. The marks ranged from 42.5% to 100%, and the average was 83.8%. There were six perfect papers. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	34.7%
A	71.5%	80-89%	36.6%
B	18.8%	70-79%	18.8%
C	7.2%	60-69%	7.2%
D	2.4%	50-59%	2.4%
F	0.3%	40-49%	0.3%
		30-39%	0.0%
		20-29%	0.0%
		10-19%	0.0 %
		0-9%	0.0%



PART I : No explanation is necessary for your answers in Question 1.

1. [avg: 8.2/10] Let

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 0 & 3 \\ 0 & 1 & -7 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be the reduced echelon form of the augmented matrix $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 & \mathbf{b} \end{bmatrix}$ of a certain system of linear equations, for vectors $\mathbf{a}_i, \mathbf{b}$ in $\mathbf{R}^4$. Answer the following questions by filling in the blanks. One mark for each correct answer.

(a) How many equations are there in the system of equations? 4

(b) How many variables are there in the system of equations? 5

(c) How many leading variables are there? 3

(d) How many free variables are there? 2

(e) Is the system of equations consistent or inconsistent? Consistent

(f) Does the system of equations have a unique solution? No

(g) Does the system of equations have infinitely many solutions? Yes

(h) Is the trivial solution a solution to the system of equations? No

(i) Is it true that span $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\} = \mathbf{R}^4$? No

(j) Is it true that $\mathbf{b} = 3\,\mathbf{a}_1 + 4\,\mathbf{a}_2 + 5\,\mathbf{a}_5$? Yes

PART II : Present **COMPLETE** solutions to the following questions in the space provided.

2. [avg: 8.6/10] For parameters s, t and u, the general solution to a linear system of equations is given by

$$x_1 = 5 + 7t - 6u,\ x_2 = s,\ x_3 = -1 - 4t + 3u,\ x_4 = t,\ x_5 = u.$$

(a) [4 marks] Express the solution as a linear combination of four (column) vectors in $\mathbf{R}^5$.

Solution:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5+7t-6u \\ s \\ -1-4t+3u \\ t \\ u \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -6 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

(b) [3 marks] Is the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 7 \\ 2 \\ 3 \end{bmatrix}$ a solution to this system?

Solution: No: $s = 1, t = 2, u = 3$ implies that $x_1 = 5 + 7t - 6u = 5 + 14 - 18 = 1$, but

$$x_3 = -1 - 4t + 3u = -1 - 8 + 9 = 0 \neq 7.$$

(c) [3 marks] Find all solutions to this system for which $x_1 = 6$ and $x_3 = 0$.

Solution: setting $x_1 = 6, x_3 = 0$ results in a system of two equations for t and u:

$$\left\{ \begin{array}{rcrcrcr} 5 & + & 7t & - & 6u & = & 6 \\ -1 & - & 4t & + & 3u & = & 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcrcr} 7t & - & 6u & = & 1 \\ -4t & + & 3u & = & 1 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rcrcr} 7t & - & 6u & = & 1 \\ -8t & + & 6u & = & 2 \end{array} \right.,$$

for which the solution is $t = -3$ and $u = -11/3$. Thus

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ s \\ 0 \\ -3 \\ -11/3 \end{bmatrix},$$

for parameter s.

3. [avg: 9.2/10] Consider the system of equations

$$\left\{\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & x_3 & + & 6x_4 & = & 0 \\ x_1 & + & 2x_2 & - & x_3 & - & 2x_4 & = & 4 \\ 2x_1 & + & 4x_2 & & & + & 4x_4 & = & 4 \end{array}\right.$$

(a) [3 marks] What is the augmented matrix of this system?

Solution:

$$\left[\begin{array}{rrrr|r} 1 & 2 & 1 & 6 & 0 \\ 1 & 2 & -1 & -2 & 4 \\ 2 & 4 & 0 & 4 & 4 \end{array}\right]$$

(b) [4 marks] Find the reduced echelon form of the augmented matrix of this system.

Solution: use elementary row operations:

$$\underbrace{\left[\begin{array}{rrrr|r} 1 & 2 & 1 & 6 & 0 \\ 1 & 2 & -1 & -2 & 4 \\ 2 & 4 & 0 & 4 & 4 \end{array}\right]}_{\text{augmented matrix}} \sim \left[\begin{array}{rrrr|r} 1 & 2 & 1 & 6 & 0 \\ 0 & 0 & -2 & -8 & 4 \\ 0 & 0 & -2 & -8 & 4 \end{array}\right] \sim \underbrace{\left[\begin{array}{rrrr|r} 1 & 2 & 0 & 2 & 2 \\ 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{reduced echelon form}}$$

(c) [3 marks] What is the solution to this system?

Solution: let $x_2 = s, x_4 = t$ be parameters. Then

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2-2s-2t \\ s \\ -2-4t \\ t \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix}}_{\text{optional form of answer}}.$$

4. [avg: 9.7/10] Balance the chemical reaction

$$\mathrm{HCl} + \mathrm{Na_3PO_4} \longrightarrow \mathrm{H_3PO_4} + \mathrm{NaCl}$$

which describes hydrochloric acid combining with sodium phosphate to produce phosphoric acid and sodium chloride, by first setting up and then solving an appropriate homogeneous system of equations. (Solving this problem by trial and error will not count!)

Solution: let

$$x_1\mathrm{HCl} + x_2\mathrm{Na_3PO_4} \longrightarrow x_3\mathrm{H_3PO_4} + x_4\mathrm{NaCl}$$

and equate the number of each type of atom on each side of the formula:

$$\begin{aligned}
\text{H:} &\quad x_1 = 3x_3 \Leftrightarrow x_1 - 3x_3 = 0 \\
\text{Cl:} &\quad x_1 = x_4 \Leftrightarrow x_1 - x_4 = 0 \\
\text{Na:} &\quad 3x_2 = x_4 \Leftrightarrow 3x_2 - x_4 = 0 \\
\text{P:} &\quad x_2 = x_3 \Leftrightarrow x_2 - x_3 = 0 \\
\text{O:} &\quad 4x_2 = 4x_3 \Leftrightarrow 4x_2 - 4x_3 = 0
\end{aligned}$$

Now solve this homogeneous system. One way is to reduce the augmented matrix:

$$\left[\begin{array}{rrrr|r} 1 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 4 & -4 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_4 = t$, then

$$x_1 = t,\ x_2 = \frac{t}{3},\ x_3 = \frac{t}{3}.$$

Find the smallest whole number solution by letting $t = 3$:

$$x_1 = 3,\ x_2 = 1,\ x_3 = 1, x_4 = 3.$$

That is, the balanced chemical reaction is:

$$3\,\mathrm{HCl} + \mathrm{Na_3PO_4} \longrightarrow \mathrm{H_3PO_4} + 3\,\mathrm{NaCl}$$

5. [avg: 8.3/10] A bacteriologist places three types of bacteria, called B_1, B_2, B_3, into a culture dish with three types of nutrients, called N_1, N_2, N_3. It is known that in a 24-hour period

- each bacterium of type B_1 will consume 4 units of N_1, 3 units of N_2, and 7 units of N_3
- each bacterium of type B_2 will consume 2 units of N_1, 1 unit of N_2, and 5 units of N_3
- each bacterium of type B_3 will consume 6 units of N_1, 2 units of N_2, and 2 units of N_3

How many bacteria of each type can be supported daily by 4200 units of N_1, 1900 units of N_2, and 4700 units of N_3?

Solution: let x_i be the number of bacteria B_i. Then the daily consumption of nutrients is given by the system of equations

$$\begin{cases} 4x_1 + 2x_2 + 6x_3 = 4200 \\ 3x_1 + x_2 + 2x_3 = 1900 \\ 7x_1 + 5x_2 + 2x_3 = 4700 \end{cases}$$

Solve this system by reducing the augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & 2 & 6 & 4200 \\ 3 & 1 & 2 & 1900 \\ 7 & 5 & 2 & 4700 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2300 \\ 3 & 1 & 2 & 1900 \\ 7 & 5 & 2 & 4700 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2300 \\ 0 & -2 & -10 & -5000 \\ 0 & -2 & -26 & -11400 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2300 \\ 0 & 1 & 5 & 2500 \\ 0 & 0 & -16 & -6400 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 4 & 2300 \\ 0 & 1 & 5 & 2500 \\ 0 & 0 & 1 & 400 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 200 \\ 0 & 1 & 0 & 500 \\ 0 & 0 & 1 & 400 \end{array}\right].$$

So 200 bacteria of type B_1, 500 of type of B_2 and 400 of type B_3 can be supported daily by the given nutrients.

6. [avg: 8.7/10] Find a set of vectors $\{\mathbf{u}, \mathbf{v}\}$ in $\mathbf{R}^4$ that spans the solution set of the system of equations

$$\begin{cases} x_1 - x_2 - 4x_3 - 4x_4 = 0 \\ 2x_1 + 2x_2 - 4x_3 + 12x_4 = 0 \\ 3x_1 - 7x_2 - 16x_3 - 32x_4 = 0 \end{cases}$$

Solution: first solve the system; then express the solution as a linear combination of two vectors. Reduce the augmented matrix of the system:

$$\left[\begin{array}{rrrr|r} 1 & -1 & -4 & -4 & 0 \\ 2 & 2 & -4 & 12 & 0 \\ 3 & -7 & -16 & -32 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -1 & -4 & -4 & 0 \\ 0 & 4 & 4 & 20 & 0 \\ 0 & -4 & -4 & -20 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & -1 & -4 & -4 & 0 \\ 0 & 1 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} 1 & 0 & -3 & 1 & 0 \\ 0 & 1 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_3 = s, x_4 = t$ be parameters. Then if $\mathbf{x}$ is a solution to this system,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3s - t \\ -s - 5t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -5 \\ 0 \\ 1 \end{bmatrix}.$$

Thus every solution to this system of equations is a combination of

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} -1 \\ -5 \\ 0 \\ 1 \end{bmatrix},$$

so $\{\mathbf{u}, \mathbf{v}\}$ is a spanning set of the set of solutions to the system.

7. [avg; 8.5/10] Consider the system of equations

$$\left\{\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & - & x_3 & + & 4x_4 & = & a \\ x_1 & - & 5x_2 & - & 4x_3 & + & 7x_4 & = & b \\ 3x_1 & + & 8x_2 & + & 11x_3 & + & k\,x_4 & = & c \end{array}\right.$$

(a) [5 marks] Find A, $\mathbf{x}$, and $\mathbf{b}$ such that $A\,\mathbf{x} = \mathbf{b}$ corresponds to this system of equations.

Solution:

$$A = \begin{bmatrix} 1 & -2 & -1 & 4 \\ 1 & -5 & -4 & 7 \\ 3 & 8 & 11 & k \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

(b) [5 marks] For which values of k do the columns of A span $\mathbf{R}^3$?

Solution: need to find values of k for which the equation $A\,\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{x}$ for every possible vector $\mathbf{b}$. To this end reduce the augmented matrix $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$:

$$\left[\begin{array}{rrrr|c} 1 & -2 & -1 & 4 & a \\ 1 & -5 & -4 & 7 & b \\ 3 & 8 & 11 & k & c \end{array}\right] \sim \left[\begin{array}{rrrc|c} 1 & -2 & -1 & 4 & a \\ 0 & -3 & -3 & 3 & b-a \\ 0 & 14 & 14 & k-12 & c-3a \end{array}\right]$$

$$\sim \left[\begin{array}{rrrc|c} 1 & -2 & -1 & 4 & a \\ 0 & 1 & 1 & -1 & (a-b)/3 \\ 0 & 0 & 0 & k+2 & c-3a-14(a-b)/3 \end{array}\right]$$

Thus:

- if $k = -2$ the system will not have a solution, if $c - 3a - 14(a-b)/3 \neq 0$,
- if $k \neq -2$ the system will have a solution, infinitely many solutions actually, no matter what a, b, c are.

So the answer to the question is: $k \neq -2$.

8. [avg: 5.8/10] Indicate if the following statements are **True** or **False**, and give a *brief* explanation why.

(a) (2 marks) If a homogeneous system of linear equations is also a triangular system, then the only solution is the trivial solution. $\otimes$ **True** $\bigcirc$ **False**

Explanation: a triangular system always has a unique solution, which for a homogeneous system must be $\mathbf{x} = \mathbf{0}$.

(b) (2 marks) If the set of vectors $\{\mathbf{u}_1, \mathbf{u}_2\}$ spans $\mathbf{R}^2$, then so does the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$, for any vector $\mathbf{u}_3$ in $\mathbf{R}^2$. $\otimes$ **True** $\bigcirc$ **False**

Explanation: every vector in $\mathbf{R}^2$ can be written as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, so it can also be written as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$ just by adding $0\,\mathbf{u}_3$.

(c) (2 marks) If A is a matrix with columns that span $\mathbf{R}^n$, then $A\,\mathbf{x} = \mathbf{0}$ has nontrivial solutions.

$\bigcirc$ **True** $\otimes$ **False**

Explanation: consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The columns obviously span $\mathbf{R}^3$, but the only solution to $A\,\mathbf{x} = \mathbf{0}$, is $\mathbf{x} = \mathbf{0}$.

(d) (2 marks) Every system of linear equations with more variables than equations must have infinitely many solutions. $\bigcirc$ **True** $\otimes$ **False**

Explanation: not true for the inconsistent system $x_1 + x_2 + x_3 = 1,\ x_1 + x_2 + x_3 = 2$.

(e) (2 marks) Adding a linear combination of two distinct rows of a matrix to another row of the matrix is not an elementary row operation.

$\otimes$ **True** $\bigcirc$ **False**

Explanation: the row operation $mR_i + nR_j + R_k$ consists of *two* elementary row operations, $mR_i + R_k$ followed by $nR_j + R_k$, if $m, n \neq 0$.

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