University of Toronto

SOLUTIONS to MAT188H1F TERM TEST 1

of **Tuesday, October 13, 2009**

Duration: 90 minutes

TOTAL MARKS: 60

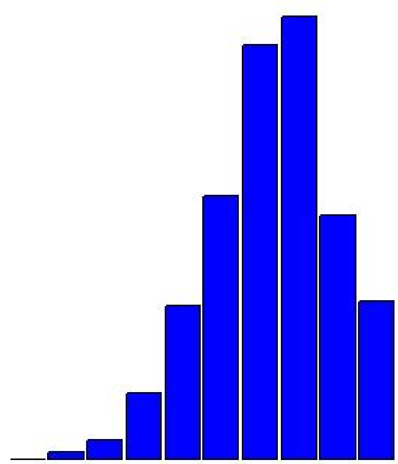
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

- Question 6, which seemed to cause the most difficulty on this test, was lifted from the book: Question 9c from Section 1.2
- Regarding Question 7: to establish that a statement is True it is NOT sufficient to give one example. You must give a general argument.
- The best way to evaluate the determinant in Question 3 is to use a combination of row or column operations and cofactor expansions. The worst way, and the most prone to error, is to simply use cofactor expansions.
- Bad notation will cost you marks! If you put $=$ or $\Rightarrow$, instead of $\rightarrow$, between reduced matrices then you should lose a mark *for each question that you do it in.*
- In Question 3 sloppy notation, like equating determinants to matrices, or vice versa, will cost you marks. A determinant is a number, and can never be equal to a matrix.

Breakdown of Results: 922 students wrote this test. The marks ranged from 15% to 100%, and the average was 67.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	8.9%
A	22.7%	80-89%	13.8%
B	25.0%	70-79%	25.0%
C	23.4%	60-69%	23.4%
D	14.9%	50-59%	14.9%
F	13.9%	40-49%	8.7%
		30-39%	3.7%
		20-29%	1.1%
		10-19%	0.4%
		0-9%	0.0%



1. [8 marks] Find the reduced row-echelon form of the matrix

$$A = \begin{bmatrix} 1 & 3 & 1 & 1 & 6 \\ 2 & 6 & 1 & -3 & 3 \\ -1 & -3 & -2 & -6 & 2 \\ 3 & 9 & 1 & -7 & 2 \end{bmatrix}.$$

What is the rank of A?

Solution:

$$\begin{bmatrix} 1 & 3 & 1 & 1 & 6 \\ 2 & 6 & 1 & -3 & 3 \\ -1 & -3 & -2 & -6 & 2 \\ 3 & 9 & 1 & -7 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 1 & 6 \\ 0 & 0 & -1 & -5 & -9 \\ 0 & 0 & -1 & -5 & 8 \\ 0 & 0 & -2 & -10 & -16 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 0 & 17 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 1 & 1 & 6 \\ 0 & 0 & 1 & 5 & 9 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 3 & 0 & -4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of A is the number of leading 1's in the reduced row-echelon matrix; so the rank is 3.

Note: The reduced row-echelon form of a matrix is unique; there is only one correct answer.

2. [8 marks] Find the inverse of the matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 4 & 2 \\ 3 & 1 & 4 \end{bmatrix}$ and use it to solve for X if $AX = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}$.

Solution:

$$(A|I) = \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 4 & 2 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 5 & 1 & -1 & 1 & 0 \\ 0 & 4 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 4 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -11 & -4 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & -1 & 0 & 12 & 4 & -5 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -11 & -4 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 14 & 5 & -6 \\ 0 & 1 & 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -11 & -4 & 5 \end{array}\right] = (I|A^{-1})$$

So

$$A^{-1} = \begin{bmatrix} 14 & 5 & -6 \\ 2 & 1 & -1 \\ -11 & -4 & 5 \end{bmatrix} \text{ and } X = A^{-1} \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 109 \\ 17 \\ -87 \end{bmatrix}.$$

Alternate Method: you could use the adjoint formula, from Section 2.2, if you wanted:

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} \operatorname{adj}(A) \\ &= \frac{1}{1} [C_{ij}(A)]^T \\ &= \begin{bmatrix} 14 & -(-2) & -11 \\ -(-5) & 1 & -(4) \\ -6 & -(1) & 5 \end{bmatrix}^T \\ &= \begin{bmatrix} 14 & 5 & -6 \\ 2 & 1 & -1 \\ -11 & -4 & 5 \end{bmatrix} \end{aligned}$$

3. [9 marks] Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & -3 & 4 & 1 \\ -1 & 1 & 2 & 2 \\ 1 & 10 & 3 & -4 \\ -1 & 2 & -1 & 1 \end{bmatrix}.$$

Solution: Use a combination of row or column operations and cofactor expansions. For example:

$$\begin{aligned}
\det A &= \det \begin{bmatrix} 2 & -3 & 4 & 1 \\ -1 & 1 & 2 & 2 \\ 1 & 10 & 3 & -4 \\ -1 & 2 & -1 & 1 \end{bmatrix} \\
&= \det \begin{bmatrix} 0 & -1 & 8 & 5 \\ -1 & 1 & 2 & 2 \\ 0 & 11 & 5 & -2 \\ 0 & 1 & -3 & -1 \end{bmatrix} \\
&= (-1)(-1)^{1+2} \det \begin{bmatrix} -1 & 8 & 5 \\ 11 & 5 & -2 \\ 1 & -3 & -1 \end{bmatrix} \\
&= \det \begin{bmatrix} -1 & 8 & 5 \\ 0 & 93 & 53 \\ 0 & 5 & 4 \end{bmatrix} \\
&= (-1)(-1)^{1+1} \det \begin{bmatrix} 93 & 53 \\ 5 & 4 \end{bmatrix} \\
&= -(93 \times 4 - 5 \times 53) \\
&= -107
\end{aligned}$$

4. [8 marks] Find all the solutions to the system of homogeneous equations

$$\begin{cases} x_1 + x_2 - 2x_3 - 3x_4 + x_5 = 0 \\ 3x_1 + 3x_2 + 2x_3 - x_4 + 11x_5 = 0 \\ -x_1 - x_2 + 6x_3 + 8x_4 + 8x_5 = 0 \end{cases}$$

and express your answer as a linear combination of basic solutions.

Solution:

$$\left[\begin{array}{rrrrr|r} 1 & 1 & -2 & -3 & 1 & 0 \\ 3 & 3 & 2 & -1 & 11 & 0 \\ -1 & -1 & 6 & 8 & 8 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 8 & 8 & 8 & 0 \\ 0 & 0 & 4 & 5 & 9 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & -3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrr|r} 1 & 1 & -2 & 0 & 16 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrrrr|r} 1 & 1 & 0 & 0 & 8 & 0 \\ 0 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{array}\right]$$

Let $x_2 = s$ and $x_5 = t$ be parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s - 8t \\ s \\ 4t \\ -5t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -8 \\ 0 \\ 4 \\ -5 \\ 1 \end{bmatrix}.$$

The solution must be expressed as a linear combination of two basic solutions to get full marks.

5. [9 marks] The two parts of this question are independent of each other.

(a) [5 marks] Let

$$A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 4 & 2 & 1 & 0 \\ -2 & 1 & 4 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -10 & 9 & -4 \\ 0 & 7 & 0 & 9 \end{bmatrix}.$$

Find elementary 3×3 matrices E and F such that $FEA = B$.

Solution: To reduce $A \to B$ two elementary row operations are used:

$$-4R_1 + R_2 \text{ and } 2R_1 + R_3.$$

The elementary matrix corresponding to the elementary operation $-4R_1 + R_2$ is

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the elementary matrix corresponding to the elementary operation $2R_1 + R_3$ is

$$F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Then $B = FEA$, although it turns out that $FE = EF$, so the order is not important.

(b) [4 marks] Suppose B is a 6×6 matrix that is obtained from the 6×6 matrix A by adding four times the second row of A to three times the fifth row of A. If $\det A = 5$ what is the value of $\det B$?

Solution: $A \to B$ via the row operations $4R_2 + 3R_5$. This is actually a combination of two elementary row operations: $3R_5$, which triples the determinant; and $4R_2 + R_5$ which does not change the determinant. Thus

$$\det B = 3 \det A = 3 \times 5 = 15.$$

6. [9 marks] Consider the system of equations with unknowns x and y :

$$\begin{cases} x + ay = 1 \\ bx + 2y = 5 \end{cases}$$

Find all the values of a and b such that the above system of equations has *(i)* no solutions, *(ii)* a unique solution, *(iii)* infinitely many solutions. What are the solutions in cases *(ii)* and *(iii)*?

Solution:

$$\left[\begin{array}{cc|c} 1 & a & 1 \\ b & 2 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & a & 1 \\ 0 & 2-ab & 5-b \end{array}\right]$$

(ii) The system has a unique solution if $2 \neq ab$, in which case the solution is

$$y = \frac{5-b}{2-ab} \text{ and } x = 1 - \frac{a(5-b)}{2-ab}.$$

(i) If $ab = 2$ but $b \neq 5$ then the system is inconsistent and has no solution.

(iii) If $ab = 2$ and $b = 5$, then $a = 2/5$ and the above reduced matrix is

$$\left[\begin{array}{cc|c} 1 & \frac{2}{5} & 1 \\ 0 & 0 & 0 \end{array}\right].$$

Let $y = t$ be a parameter. The infinitely many solutions are given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 - \frac{2}{5}t \\ t \end{bmatrix}.$$

7. [9 marks; 3 mark for each part] Determine if the following statements are True or False. Circle your choice. Justify your choice with a *brief* explanation.

(a) If the solution to a system of equations $AX = B$ is given by

$$X = \begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix},$$

for parameters s and t, then $A \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = O.$ **True** **False**

Solution: True. Based on a Theorem in Section 1.4 we know that the parametric part of a solution to the system $AX = B$ must satisfy the corresponding homogeneous system $AX = O$. OR:

$$s = 0, t = 0 \Rightarrow A \begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \end{bmatrix} = B \text{ and } s = 1, t = 0 \Rightarrow A \left(\begin{bmatrix} 3 \\ -1 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right) = B.$$

Now subtract to conclude

$$A \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = O.$$

(b) If A is an invertible matrix such that $A^2 = A^3$, then $A = I$. **True** **False**

Solution: True.

$$A^2 = A^3 \Rightarrow A^{-1}(A^2) = A^{-1}(A^3) \Rightarrow A = A^2 \Rightarrow A^{-1}A = A^{-1}A^2 \Rightarrow I = A$$

(c) Every non-zero square matrix is a product of elementary matrices. **True** **False**

Solution: False. Only *invertible* matrices are products of elementary matrices. Counter example:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq O$$

and is not a product of elementary matrices.