University of Toronto

SOLUTIONS to MAT 188H1F TERM TEST 1

of **Thursday, October 11, 2007**

Duration: 60 minutes

TOTAL MARKS: 50

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments about the Test:

1. Questions 1, 2, 3, 4 and 5(b) are completely routine calculations, making up 66% of the test. These are the parts that should allow everybody to pass this test.

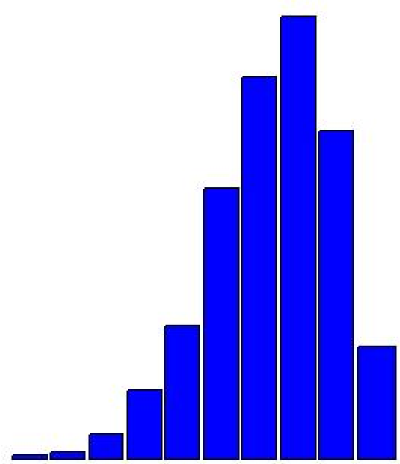
2. Question 6 is a type of question that occurs in the homework. It can be done with or without determinants. It requires some thought to set up.

3. In Question 7, two of the True or False – parts (a) and (b) – are very straightforward

4. Questions 5(a) and 7(c) require a little thought and a fair bit of computation for the marks allotted.

Breakdown of Results: 1002 students wrote this test. The marks ranged from 8% to 100%, and the average was 67.2%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	7.0%
A	25.3%%	80-89%	18.4%
B	24.8%	70-79%	24.8%
C	21.4%	60-69%	21.4%
D	15.2%	50-59%	15.2%
F	13.4%	40-49%	7.5%
		30-39%	3.9%
		20-29%	1.4%
		10-19%	0.4%
		0-9%	0.2 %



1. [7 marks] Find the reduced row-echelon form of the matrix

$$A = \begin{bmatrix} 1 & 3 & -1 & 5 \\ -2 & 4 & 3 & 0 \\ 1 & -7 & -2 & -5 \\ 2 & -4 & 5 & 0 \end{bmatrix}.$$

What is the rank of A?

Solution:

$$\begin{bmatrix} 1 & 3 & -1 & 5 \\ -2 & 4 & 3 & 0 \\ 1 & -7 & -2 & -5 \\ 2 & -4 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 10 & 1 & 10 \\ 0 & -10 & -1 & -10 \\ 0 & -10 & 7 & -10 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 10 & 1 & 10 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 10 & 0 & 10 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The rank of A is 3.

Note: The reduced row-echelon form of a matrix is unique; there is only one correct answer.

2. [8 marks] Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{bmatrix}$ and use it to solve for X if $AX = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.

Solution:

$$(A|I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & -1 & -6 & 1 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right] = (I|A^{-1})$$

So

$$A^{-1} = \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \text{ and } X = A^{-1} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -11 \\ -5 \\ 6 \end{bmatrix}.$$

Alternate Method: you could use the adjoint formula to find A^{-1} :

$$\begin{aligned} A^{-1} &= \frac{1}{\det A} \operatorname{adj}(A) \\ &= \frac{1}{1} [C_{ij}(A)]^T \\ &= \begin{bmatrix} -11 & -4 & 6 \\ -(-2) & 0 & -1 \\ 2 & -(-1) & -1 \end{bmatrix}^T \\ &= \begin{bmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{bmatrix} \end{aligned}$$

3. [7 marks] Find the determinant of the matrix

$$A = \begin{bmatrix} 1 & 2 & 7 & -5 \\ 2 & 6 & 11 & 1 \\ -1 & 0 & -6 & 13 \\ -2 & 0 & -15 & 30 \end{bmatrix}.$$

Solution: Use a combination of row or column operations and cofactor expansions. For example:

$$\begin{aligned}
\det A &= 2\det \begin{bmatrix} 1 & 1 & 7 & -5 \\ 2 & 3 & 11 & 1 \\ -1 & 0 & -6 & 13 \\ -2 & 0 & -15 & 30 \end{bmatrix} \\
&= 2\det \begin{bmatrix} 1 & 1 & 7 & -5 \\ -1 & 0 & -10 & 16 \\ -1 & 0 & -6 & 13 \\ -2 & 0 & -15 & 30 \end{bmatrix} \\
&= 2(-1)\det \begin{bmatrix} -1 & -10 & 16 \\ -1 & -6 & 13 \\ -2 & -15 & 30 \end{bmatrix} \\
&= -2\det \begin{bmatrix} -1 & -10 & 16 \\ 0 & 4 & -3 \\ 0 & 5 & -2 \end{bmatrix} \\
&= (-2)(-1)\det \begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix} \\
&= 2(-8+15) \\
&= 14
\end{aligned}$$

4. [7 marks] Find all the solutions to the system of homogeneous equations

$$\begin{cases} 4x_1 & + & 8x_2 & - & x_3 & + & 5x_4 & = & 0 \\ 2x_1 & + & 4x_2 & + & 3x_3 & - & x_4 & = & 0 \\ 2x_1 & + & 4x_2 & - & 4x_3 & + & 6x_4 & = & 0 \end{cases}$$

and express your answer as a linear combination of basic solutions.

Solution:

$$\left[\begin{array}{cccc|c} 4 & 8 & -1 & 5 & 0 \\ 2 & 4 & 3 & -1 & 0 \\ 2 & 4 & -4 & 6 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 4 & 8 & -1 & 5 & 0 \\ 0 & 0 & 7 & -7 & 0 \\ 0 & 0 & -7 & 7 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 4 & 8 & -1 & 5 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 4 & 8 & 0 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Let $x_2 = s$ and $x_4 = t$ be parameters. Then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s - t \\ s \\ t \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

The solution must be expressed as a linear combination to get full marks.

5. [7 marks] The parts of this question are unrelated.

(a) [3 marks] Express the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ as a product of elementary matrices.

Solution: Reduce $A \to I$ by interchanging R_1 and R_2, then by interchanging R_2 and R_3, and then by interchanging R_3 and R_4. Then

$$I = E_3E_2E_1A \Leftrightarrow A = E_1^{-1}E_2^{-1}E_3^{-1} \Leftrightarrow A = E_1E_2E_3,$$

for

$$E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(b) [4 marks] Suppose A and B are 3×3 matrices such that $\det A = -3$ and $\det B = 2$. Find the value of

$$\det\left(A^3B^T(2A)^{-1}B^3\right).$$

Solution:

$$\begin{aligned} \det\left(A^3B^T(2A)^{-1}B^3\right) &= \det A^3 \det B^T \det(2A)^{-1} \det B^3 \\ &= (\det A)^3 \det B \det\left(\frac{1}{2}A^{-1}\right)(\det B)^3 \\ &= (\det A)^3 \det B \left(\frac{1}{8}(\det A)^{-1}\right)(\det B)^3 \\ &= \frac{1}{8}(\det A)^2(\det B)^4 \\ &= \frac{1}{8}(-3)^2 2^4 \\ &= 18 \end{aligned}$$

6. [8 marks] For which values of a and b does the system of equations

$$\begin{cases} x + y + az = 0 \\ -x + z = 0 \\ ax - y + z = b \end{cases}$$

have *(i)* no solutions? *(ii)* a unique solution? *(iii)* infinitely many solutions?

Solution:

$$\det \begin{bmatrix} 1 & 1 & a \\ -1 & 0 & 1 \\ a & -1 & 1 \end{bmatrix} = 2 + 2a = 2(1+a).$$

(ii) The system has a unique solution if the coefficient matrix is invertible

$$\Leftrightarrow 1 + a \neq 0 \Leftrightarrow a \neq -1.$$

If $a = -1$, reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -1 & 1 & b \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b \end{array}\right]$$

(i) So if $a = -1$ and $b \neq 0$, the system has no solutions.

(iii) But if $a = -1$ and $b = 0$, the system has infinitely many solutions.

Alternate Solution: reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & a & 0 \\ -1 & 0 & 1 & 0 \\ a & -1 & 1 & b \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & a & 0 \\ 0 & 1 & 1+a & 0 \\ 0 & -1-a & 1-a^2 & b \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & a & 0 \\ 0 & 1 & 1+a & 0 \\ 0 & 0 & 2+2a & b \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1+a & 0 \\ 0 & 0 & 2+2a & b \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -b/2 \\ 0 & 0 & 2+2a & b \end{array}\right]$$

This augmented matrix corresponds to the system

$$x = z, y = -\frac{b}{2}, \text{ and } (2+2a)z = b.$$

So, *(i)* the system has no solutions if $a = -1$ and $b \neq 0$;

(ii) the system has a unique solution if $a \neq -1$;

(iii) and the system has infinitely many solutions if $a = -1$ and $b = 0$.

7. [6 marks; 2 marks for each part] Determine if the following statements are True or False. Justify your choice with a *brief* explanation.

(a) Every homogeneous system of linear equations is consistent.

True: because the trivial solution is always a solution to a homogeneous system.

(b) If A and B are 3×3 matrices, then $\det(A+B) = \det(A) + \det(B)$.

False: If

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then $\det A = 0, \det B = 0$, but $\det(A+B) = \det I = 1$.

(c) Suppose A is a 3×3 matrix such that $A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. If $\det A = 2$, then the third column of $\text{adj}(A)$ is $\begin{bmatrix} 1/2 \\ 1 \\ 3/2 \end{bmatrix}$.

False:

$$\begin{aligned}
A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad &\Rightarrow \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&\Rightarrow \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{\det A} \text{adj}(A) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&\Rightarrow \quad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \frac{1}{2} \text{adj}(A) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&\Rightarrow \quad 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \text{adj}(A) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\
&\Rightarrow \quad \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \text{third column of adj}(A)
\end{aligned}$$