MAT188H1F - Linear Algebra - Fall 2017

Solutions to Term Test 2 - November 7, 2017

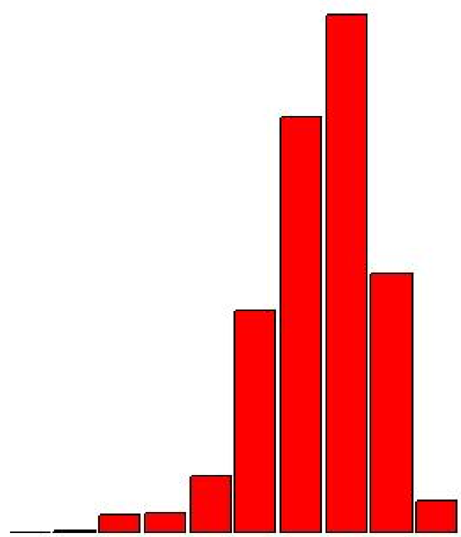
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Comments::

- Every question had a passing average, except for Question 6, which hardly an average at all! 568 students had zero on Question 6, only 23 students passed it, and just one student scored 10!
- In Question 1 many students threw a mark away for bad notation, or incorrect form of answer. If the answer for part(c) is $\{\vec{a}, \vec{b}, \vec{c}\}$ then that is *not* the same as span$\{\vec{a}, \vec{b}, \vec{c}\}$, which is a subspace. Nor is the answer row(A) = span$\{\vec{a}, \vec{b}, \vec{c}\}$, which is just another way of writing a subspace. Nor is the answer row$(A) = \{\vec{a}, \vec{b}, \vec{c}\}$, which is very bad, since a subspace can't equal a set of three vectors. (Similar abuse of notation occurred in other questions, and also resulted in marks being deducted.)

Breakdown of Results: 819 of 826 registered students wrote this test. The marks ranged from 18.8% to 97.5%, and the average was 68.6%. (Without Question 6, the average was 77.4%) Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.1%
A	18.7%	80-89%	16.6%
B	33.7%	70-79%	33.7%
C	27.1%	60-69%	27.1%
D	14.4%	50-59%	14.4%
F	6.1%	40-49%	3.7%
		30-39%	1.2%
		20-29%	1.1%
		10-19%	0.1%
		0-9%	0.0%



1. [avg: 9.0/10] Given that the reduced row echelon form of

$$A = \begin{bmatrix} 1 & 3 & -1 & 7 & -6 \\ 5 & 15 & 4 & 5 & 27 \\ 3 & 9 & 6 & -9 & 39 \\ -6 & -18 & 2 & 5 & -23 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the following. (No explanations are required for this question; but explanations given.)

(a) [1 mark] The rank of A. **Answer:** the number of leading 1's in R : 3

(b) [1 mark] The nullity of A. **Answer:** the number of cols of A minus its rank: 2

(c) [3 marks] A basis for the row space of A. **Answer:** the non-zero rows of R,

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\},$$

or any three **independent** rows of A.

(d) [3 marks] A basis for the column space of A. **Answer:** the cols of A that correspond to the cols of R with leading 1's:

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 3 \\ -6 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ -9 \\ 5 \end{bmatrix} \right\},$$

or any three **independent** cols of A.

(e) [2 marks] A basis for the null space of A. Use $A\vec{x} = \vec{0} \Leftrightarrow R\vec{x} = \vec{0}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3s - 4t \\ s \\ -3t \\ t \\ t \end{bmatrix}; \text{ so the } \textbf{answer} \text{ is: } \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 1 \end{bmatrix} \right\}$$

2. [avg: 9.4/10] A student is taking courses in algebra, calculus and physics at a college where grades are given in percentages. To determine her standing for a physics prize, a weighted average is calculated based on 50% of her physics grade, 30% of her calculus grade and 20% of her algebra grade: the weighted average is 91.2%. For an applied mathematics prize, a weighted average based on one-third of each of the three grades is calculated to be 90%. For a pure mathematics prize, her average based on 50% of her calculus grade and 50% of her algebra grade is 87%. What are her grades in the three individual courses? **Note:** to do well on this question you must define your variables and you must write down the system of linear equations that has to be solved, but you can solve it using any method you like, as long as you show your work.

Solution: let her algebra mark be a, let her calculus mark be c, let her physics mark be p. Then we have

$$\left.\begin{array}{rcrcrcl} \frac{1}{2}p & + & \frac{3}{10}c & + & \frac{1}{5}a & = & 91.2 \\ \frac{1}{3}p & + & \frac{1}{3}c & + & \frac{1}{3}a & = & 90 \\ & & \frac{1}{2}c & + & \frac{1}{2}a & = & 87 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{rcrcrcl} 5p & + & 3c & + & 2a & = & 912 \\ p & + & c & + & a & = & 270 \\ & & c & + & a & = & 174 \end{array}\right.$$

Subtracting the third equation from the second immediately gives us $p = 96$. Then we have two equations for a and c :

$$c + a = 174 \text{ and } 3c + 2a = 912 - 5(96) = 432.$$

Subtracting two times the left one from the right one gives $c = 432 - 2(174) = 84$. Finally,

$$c + a = 174 \Rightarrow a = 174 - c = 90.$$

Thus her marks are 90% in algebra, 84% in calculus, and 96% in physics.

Using row reduction on an augmented matrix:

$$\left[\begin{array}{ccc|c} 5 & 3 & 2 & 912 \\ 1 & 1 & 1 & 270 \\ 0 & 1 & 1 & 174 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 270 \\ 0 & 1 & 1 & 174 \\ 5 & 3 & 2 & 912 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 270 \\ 0 & 1 & 1 & 174 \\ 0 & 2 & 3 & 438 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 270 \\ 0 & 1 & 1 & 174 \\ 0 & 0 & 1 & 90 \end{array}\right],$$

from which backward substitution gives $a = 90,\ c = 84$ and $p = 96$.

3. [avg: 9.1/10] Let $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear transformation defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + x_2 \\ -2x_1 + x_2 \end{bmatrix}.$$

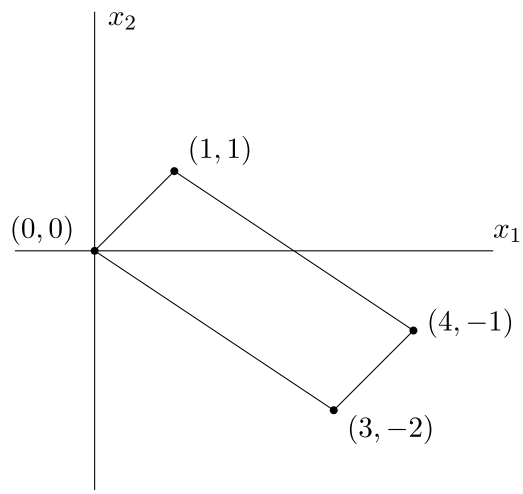
(a) [5 marks] Let $\mathcal{U}$ be the square with vertices $(0,0), (1,0), (0,1), (1,1)$. Find the image under L of each vertex of the square, and then sketch the image of $\mathcal{U}$ under L. Label the vertices!

Solution: the image of the unit square is the parallelogram determined by the origin and the points

$$L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix},$$

$$L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$



(b) [5 marks] Find $L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$.

Solution: from part (a), the matrix of L is $[L] = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$. Then $[L^{-1}] = [L]^{-1}$;

$$\left[\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ -2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & 5 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 5 & 0 & 1 & -1 \\ 0 & 5 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1/5 & -1/5 \\ 0 & 1 & 2/5 & 3/5 \end{array}\right]$$

So

$$[L]^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

OR simply use the formula for the inverse of a 2×2 matrix:

$$[L]^{-1} = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}^{-1} = \frac{1}{3+2}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}.$$

Either way,

$$L^{-1}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ or } \frac{1}{5}\begin{bmatrix} x_1 - x_2 \\ 2x_1 + 3x_2 \end{bmatrix}$$

4. [avg: 8.3/10] Consider the system of equations $(*)\ \begin{cases} x_1 + 4x_2 + x_3 = 2 \\ 3x_1 - 2x_2 + 2x_3 = -4 \\ x_1 + 2x_2 + x_3 = 6 \end{cases}$.

(a) [2 marks] Write this system as a single matrix equation $A\vec{x} = \vec{b}$.

Solution: $\begin{bmatrix} 1 & 4 & 1 \\ 3 & -2 & 2 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$

(b) [5 marks] Find A^{-1}. **Solution** using the Gaussian algorithm:

$$\left[\begin{array}{c|c} A & I \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 3 & -2 & 2 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & -14 & -1 & -3 & 1 & 0 \\ 0 & -2 & 0 & -1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & -1 \\ 0 & 14 & 1 & 3 & -1 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 & -1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 5 & 1 & -7 \\ 0 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 & -1 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -5 \\ 0 & 2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 & -1 & 7 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & -5 \\ 0 & 1 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 1 & -4 & -1 & 7 \end{array}\right] = \left[\begin{array}{c|c} I & A^{-1} \end{array}\right];\ \text{that is, } A^{-1} = \begin{bmatrix} 3 & 1 & -5 \\ 1/2 & 0 & -1/2 \\ -4 & -1 & 7 \end{bmatrix}.$$

(c) [3 marks] Use A^{-1} to solve the system of equations $(*)$.

Soluiton: $A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$, so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -5 \\ 1/2 & 0 & -1/2 \\ -4 & -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix} = \begin{bmatrix} 6-4-30 \\ 1+0-3 \\ -8+4+42 \end{bmatrix} = \begin{bmatrix} -28 \\ -2 \\ 38 \end{bmatrix}$$

5.lavg: 6.8/10] Recall that the reflection in the plane with normal vector $\vec{n}$ is given by

$$\operatorname{refl}_{\vec{n}}(\vec{x}) = \vec{x} - 2\operatorname{proj}_{\vec{n}}(\vec{x}).$$

Let $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$; let $A = [\operatorname{refl}_{\vec{n}}]$ be the standard matrix of $\operatorname{refl}_{\vec{n}}$.

(a) [6 marks] Find A. **Solution:** let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ be the standard basis of $\mathbb{R}^3$. We have $\|\vec{n}\|^2 = 6$ and

$$\left.\begin{aligned}
\operatorname{refl}_{\vec{n}}(\vec{e}_1) &= \vec{e}_1 - 2\operatorname{proj}_{\vec{n}}(\vec{e}_1) = \vec{e}_1 - 2\left(\frac{\vec{e}_1\cdot\vec{n}}{\|\vec{n}\|^2}\right)\vec{n} = \vec{e}_1 - \frac{1}{3}\vec{n} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \\
\operatorname{refl}_{\vec{n}}(\vec{e}_2) &= \vec{e}_2 - 2\operatorname{proj}_{\vec{n}}(\vec{e}_2) = \vec{e}_2 - 2\left(\frac{\vec{e}_2\cdot\vec{n}}{\|\vec{n}\|^2}\right)\vec{n} = \vec{e}_2 + \frac{1}{3}\vec{n} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \\
\operatorname{refl}_{\vec{n}}(\vec{e}_3) &= \vec{e}_3 - 2\operatorname{proj}_{\vec{n}}(\vec{e}_3) = \vec{e}_3 - 2\left(\frac{\vec{e}_3\cdot\vec{n}}{\|\vec{n}\|^2}\right)\vec{n} = \vec{e}_3 - \frac{2}{3}\vec{n} = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}
\end{aligned}\right\} \Rightarrow A = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix}$$

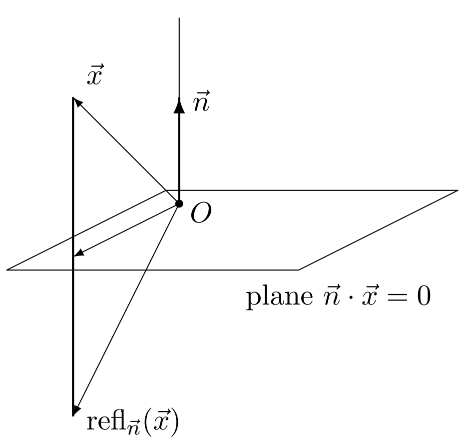
(b) [2 marks] Verify algebraically that $A^{-1} = A$. Show your calculations!

Solution: show that $A\,A = I$:

$$\frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & 2 \\ -2 & 2 & -1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4+1+4 & 2+2-4 & -4+2+2 \\ 2+2-4 & 1+4+4 & -2+4-2 \\ -4+2+2 & -2+4-2 & 4+4+1 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = I$$

(c) [2 marks] Explain geometrically why $A^{-1} = A$.

Solution:



By symmetry,

$$\operatorname{refl}_{\vec{n}}(\operatorname{refl}_{\vec{n}}(\vec{x})) = \vec{x};$$

that is, $\operatorname{refl}_{\vec{n}} \circ \operatorname{refl}_{\vec{n}} = \mathrm{Id}$. In terms of matrices, this means

$$A^2 = I \Leftrightarrow A^{-1} = A.$$

6. [avg: 0.7/10] Let A be an $m \times k$ matrix; let B be a $k \times n$ matrix.

(a) [3 marks] Show that $\text{nullity}(B) \leq \text{nullity}(A\,B)$.

Solution: show $\text{null}(B)$ is contained in $\text{null}(A\,B)$, and the result follows by taking dimensions.

$$\begin{aligned} \vec{x} \in \text{null}(B) \quad &\Rightarrow \quad B(\vec{x}) = \vec{0} \\ &\Rightarrow \quad A(B(\vec{x})) = A(\vec{0}) = \vec{0} \\ &\Rightarrow \quad x \in \text{null}(A\,B) \end{aligned}$$

Then $\dim(\text{null}(B)) \leq \dim(\text{null}(A\,B)$; or equivalently, $\text{nullity}(B) \leq \text{nullity}(A\,B)$.

(b) [3 marks] Show that $\text{rank}(A\,B) \leq \text{rank}(A)$.

Solution: show $\text{col}(AB)$ is contained in $\text{col}(A)$, and the result follows by taking dimensions.

$$\begin{aligned} \vec{y} \in \text{col}(AB) \quad &\Rightarrow \quad \vec{y} = (AB)\vec{x}, \text{or some } \vec{x} \in \mathbb{R}^n \\ . \quad &\Rightarrow \quad \vec{y} = A(B\vec{x}) = A\vec{z}, \text{ for } \vec{z} = B\vec{x} \in \mathbb{R}^k \\ &\Rightarrow \quad \vec{y} \in \text{col}(A). \end{aligned}$$

Then

$$\text{rank}(A\,B) = \dim(\text{col}(AB)) \leq \dim(\text{col}(A)) = \text{rank}(A).$$

(c) [4 marks; 2 marks for each part] Show that:

- if AA^T is invertible then $\text{rank}(A) = m$.

  Solution: observe that A^T is a $k \times m$ matrix. If AA^T is invertible then $\text{nullity}(AA^T) = 0$. Thus, making use of part (a):

  $$\text{nullity}(A^T) \leq \text{nullity}(AA^T) = 0 \Rightarrow \text{nullity}(A^T) = 0 \Rightarrow \text{rank}(A^T) = m$$

  Conclusion: $\text{rank}(A) = \text{rank}(A^T) = m$.

- if $A^T A$ is invertible then $\text{rank}(A) = k$.

  Solution: if $A^T A$ is invertible then $\text{nullity}(A^T A) = 0$. Thus, making use of part (a):

  $$\text{nullity}(A) \leq \text{nullity}(A^T A) = 0 \Rightarrow \text{nullity}(A) = 0 \Rightarrow \text{rank}(A) = k.$$

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See Page 10 for some alternate solutions to parts of Question 6.

7. [avg: 5.2/10] Suppose $L : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear transformation. A vector $\vec{x} \in \mathbb{R}^n$ is called a **fixed point** of L if $L(\vec{x}) = \vec{x}$. Let $\mathcal{F}$ be the set of fixed points of L; that is, in set notation, $\mathcal{F} = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = \vec{x}\}$.

(a) [4 marks] Show that $\mathcal{F}$ is a subspace of $\mathbb{R}^n$.

Solution: we have $L(\vec{x}) = \vec{x} \Leftrightarrow (L - \mathrm{Id})(\vec{x}) = \vec{0}$; that is, $\mathcal{F} = \mathrm{null}(L - Id)$, or

$$\mathcal{F} = \mathrm{null}([L - \mathrm{Id}]).$$

OR you could use the definition of subspace:

1. $\mathcal{F}$ is non-empty: $L(\vec{0}) = \vec{0} \Rightarrow \vec{0} \in \mathcal{F}$
2. $\mathcal{F}$ is closed under addition: if vectors $\vec{x}, \vec{y}$ are in $\mathcal{F}$ then

$$L(\vec{x} + \vec{y}) = L(\vec{x}) + L(\vec{y}) = \vec{x} + \vec{y},$$

so $\vec{x} + \vec{y} \in \mathcal{F}$.

3. if k is a scalar and $\vec{x} \in \mathcal{F}$, then $L(k\vec{x}) = kL(\vec{x}) = k\,\vec{x}$, so $k\,\vec{x} \in \mathcal{F}$.

(b) [6 marks; 2 marks for each part.] Let $\vec{v}$ be a non-zero vector in $\mathbb{R}^3$. Describe geometrically the subspace of fixed points for each of the following linear transformations $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$.

(*i*) $L(\vec{x}) = \mathrm{proj}_{\vec{v}}\,(\vec{x})$

Solution: $\mathrm{proj}_{\vec{v}}\,(\vec{x}) = \vec{x} \Leftrightarrow \left(\dfrac{\vec{x} \cdot \vec{v}}{\|\vec{v}\|^2}\right) \vec{v} = \vec{x}$; so $\mathcal{F}$ is the line $\mathrm{span}\{\vec{v}\}$, that is, the line with vector equation $\vec{x} = t\,\vec{v}$.

(*ii*) $L(\vec{x}) = \mathrm{perp}_{\vec{v}}\,(\vec{x})$

Solution: $\mathrm{perp}_{\vec{v}}\,\vec{x} = \vec{x} \Leftrightarrow \vec{x} - \mathrm{proj}_{\vec{v}}(\vec{x}) = \vec{x} \Leftrightarrow \mathrm{proj}_{\vec{v}}(\vec{x}) = \vec{0} \Leftrightarrow \vec{x} \cdot \vec{v} = 0$; so $\mathcal{F}$ is the plane with equation $\vec{v} \cdot \vec{x} = 0$.

(*iii*) $L(\vec{x}) = \mathrm{refl}_{\vec{v}}\,(\vec{x})$

Solution: $\mathrm{refl}_{\vec{v}}\,(\vec{x}) = \vec{x} \Leftrightarrow \vec{x} - 2\mathrm{proj}_{\vec{v}}(\vec{x}) = \vec{x} \Leftrightarrow \mathrm{proj}_{\vec{v}}(\vec{x}) = \vec{0} \Leftrightarrow \vec{x} \cdot \vec{v} = 0$; and again, $\mathcal{F}$ is the plane with equation $\vec{v} \cdot \vec{x} = 0$.

8. [avg: 6.3/10] Indicate if the following statements are **True** or **False**, and give a *brief* explanation why.

(a) [2 marks] If A and B are 2×2 matrices, then $(A+B)^2 = A^2 + 2AB + B^2$. ○ **True** ⊗ **False**

Reason: $(A+B)^2 = (A+B)(A+B) = A^2 + BA + AB + B^2$. Since matrix multiplication is not commutative, $AB \neq BA$ in general. For example, the statement is False if

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \text{ since } AB = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 2 & 7 \\ 0 & 1 \end{bmatrix}$$

(b) [2 marks] If $R = \begin{bmatrix} \cos(\pi/60) & -\sin(\pi/60) \\ \sin(\pi/60) & \cos(\pi/60) \end{bmatrix}$ then $R^{90} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. ⊗ **True** ○ **False**

Reason: R is a rotation of $\pi/60$, so R^{90} is a rotation of $90 \times \pi/60 = 3\pi/2$, which has matrix

$$\begin{bmatrix} \cos(3\pi/2) & -\sin(3\pi/2) \\ \sin(3\pi/2) & \cos(3\pi/2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(c) [2 marks] $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 8 \end{bmatrix} \right\}$ is linearly dependent. ⊗ **True** ○ **False**

Reason: in $\mathbb{R}^3$ any set of four or more vectors is linearly dependent.

(d) [2 marks] $\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ 6 \\ 4 \\ 5 \end{bmatrix} \right\}$ is linearly independent. ○ **True** ⊗ **False**

Reason: the third vector is the second vector minus the first, so they are dependent.

(e) [2 marks] If A is an 8×5 matrix with $\text{rank}(A) = 4$, then the columns of A are linearly dependent.

⊗ **True** ○ **False**

Reason: $\dim(\text{col}(A)) = 4$, so any five vectors in $\text{col}(A)$ must be linearly dependent.

Some alternate solutions to Question 6:

Note: If you use the Rank-Nullity Theorem and the result from part (a), you can derive a similar result to (b), which is equally useful for doing part (c). $A\,B$ is an $m \times n$ matrix and

$$\begin{aligned} \text{nullity}(B) \le \text{nullity}(A\,B) \quad &\Rightarrow \quad n - \text{rank}(B) \le n - \text{rank}(A\,B) \\ &\Rightarrow \quad \text{rank}(A\,B) \le \text{rank}(B). \end{aligned}$$

For part(c):

- if AA^T is invertible then $\text{rank}(A) = m$.

  Solution: observe that AA^T is an $m \times m$ matrix; if it is invertible then $\text{rank}(AA^T) = m$. Thus, making use of part (b):

  $$m = \text{rank}(AA^T) \le \text{rank}(A) \le m, \text{ the number of rows of } A.$$

  Conclusion: $\text{rank}(A) = m$.

- if A^TA is invertible then $\text{rank}(A) = k$.

  Solution: observe that A^TA is a $k \times k$ matrix; if it is invertible then $\text{rank}(A^TA) = k$. Thus, making use of part (b):

  $$k = \text{rank}(A^TA) \le \text{rank}(A^T) = \text{rank}(A) \le k, \text{ the number of cols of } A.$$

  Conclusion: $\text{rank}(A) = k$.

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The end.