University of Toronto
Solutions to **MAT188H1F TERM TEST**
of **Tuesday, November 1, 2011**
Duration: 110 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

INSTRUCTIONS: Answer all questions. Present your solutions in the space provided. Use the back of the page if you need more space. Make sure this test contains 9 pages. Do not tear any pages from this test. The value for each question is indicated in parantheses beside the question number. **TOTAL MARKS: 70**

General Comments about the Test:

1. The coefficient matrix for Question 3 came from Exercises 2.2 #14(b), which asked for the values of c for which the matrix has an inverse. That calculation is the key to solving this problem, but many students didn't make the connection.

2. Whenever you solve for eigenvalues and eigenvectors, as in #4 and #7, your 'eigenvector' will be the zero vector if your eigenvalue is incorrect. If this happens, you should go back and find the correct eigenvalues.

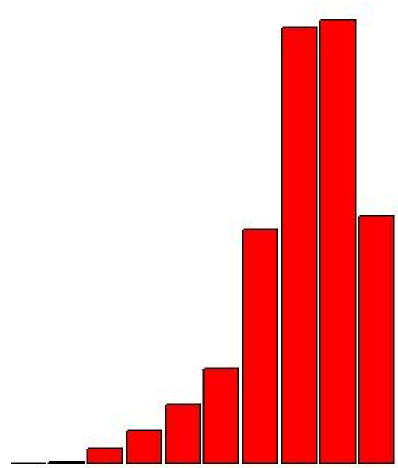
3. In #6 many students did not know, or did not use, the projection formula. The projection formula, generalized to n dimensions, will be used repeatedly in parts of Chapter 4; make sure you know it.

4. To show a statement is true it is not enough to give an example; you must give a general argument–a proof.

5. There were 19 perfect papers.

Breakdown of Results: 960 students wrote this test. The marks ranged from 15.7% to 100%, and the average was 75.3%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	15.9%
A	44.3%	80-89%	28.4%
B	27.9%	70-79%	27.9%
C	14.9%	60-69%	14.9%
D	6.0%	50-59%	6.0%
F	6.9%	40-49%	3.8%
		30-39%	2.1%
		20-29%	0.9%
		10-19%	0.1%
		0-9%	0.0%



1. [8 marks; 4 marks for each part] Compute the following:

(a) $\det \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & 5 & 4 \\ -1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 1 \end{bmatrix}$

Solution: use properties of determinants to simplify your work.

$$\det \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & 5 & 4 \\ -1 & 2 & 1 & 3 \\ 2 & 3 & 2 & 1 \end{bmatrix} = \det \begin{bmatrix} 0 & 4 & 0 & 4 \\ 0 & 8 & 7 & 10 \\ -1 & 2 & 1 & 3 \\ 0 & 7 & 4 & 7 \end{bmatrix} = (-1)(-1)^4 \det \begin{bmatrix} 4 & 0 & 4 \\ 8 & 7 & 10 \\ 7 & 4 & 7 \end{bmatrix}$$

$$= -\det \begin{bmatrix} 4 & 0 & 0 \\ 8 & 7 & 2 \\ 7 & 4 & 0 \end{bmatrix} = -4 \det \begin{bmatrix} 7 & 2 \\ 4 & 0 \end{bmatrix} = (-4)(-8) = 32$$

(b) $\operatorname{adj} \begin{bmatrix} 1 & 1 & 2 \\ -1 & 7 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution: need the 9 cofactors.

$$C_{11} = \det \begin{bmatrix} 7 & 1 \\ -1 & 3 \end{bmatrix} = 22 \qquad C_{12} = -\det \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} = 5 \qquad C_{13} = \det \begin{bmatrix} -1 & 7 \\ 2 & -1 \end{bmatrix} = -13$$

$$C_{21} = -\det \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = -5 \qquad C_{22} = \det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = -1 \qquad C_{23} = -\det \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} = 3$$

$$C_{31} = \det \begin{bmatrix} 1 & 2 \\ 7 & 1 \end{bmatrix} = -13 \qquad C_{32} = -\det \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = -3 \qquad C_{33} = \det \begin{bmatrix} 1 & 1 \\ -1 & 7 \end{bmatrix} = 8$$

So

$$\operatorname{adj} \begin{bmatrix} 1 & 1 & 2 \\ -1 & 7 & 1 \\ 2 & -1 & 3 \end{bmatrix} = [Cij]^T = \begin{bmatrix} 22 & -5 & -13 \\ 5 & -1 & -3 \\ -13 & 3 & 8 \end{bmatrix}$$

2(a). [4 marks] Find the value of x_2 in the solution to the system of equations

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & - & x_3 & = & 5 \\ 2x_1 & - & x_2 & + & 3x_3 & = & 2 \\ -x_1 & + & x_2 & + & 2x_3 & = & 1 \end{array}$$

Solution: use Cramer's Rule.

$$x_2 = \frac{\det \begin{bmatrix} 1 & 5 & -1 \\ 2 & 2 & 3 \\ -1 & 1 & 2 \end{bmatrix}}{\det \begin{bmatrix} 1 & 3 & -1 \\ 2 & -1 & 3 \\ -1 & 1 & 2 \end{bmatrix}} = \frac{4 - 15 - 2 - 2 - 3 - 20}{-2 - 9 - 2 + 1 - 3 - 12} = \frac{38}{27}$$

2(b). [4 marks] If A is a 4×4 matrix and $\det(A) = -2$, what is the value of $\det(\text{adj}(A))$?

Solution: use the adjoint formula, and the properties of the determinant function.

$$\begin{aligned} A\ \text{adj}(A) = \det(A)\, I \quad & \Rightarrow \quad A\ \text{adj}(A) = -2I \\ & \Rightarrow \quad \det(A\ \text{adj}(A)) = \det(-2I) = (-2)^4 = 16 \\ & \Rightarrow \quad \det(A) \det(\text{adj}(A)) = 16 \\ & \Rightarrow \quad \det(\text{adj}(A)) = 16/(-2) = -8 \end{aligned}$$

3. [8 marks] For which values of c does the following system of equations have *(i)* no solutions, *(ii)* a unique solution, *(iii)* infinitely many solutions?

$$\begin{array}{rcrcrcl} x_1 & - & c\,x_2 & + & c\,x_3 & = & 4 \\ x_1 & + & x_2 & - & x_3 & = & 2 \\ c\,x_1 & - & c\,x_2 & + & x_3 & = & 4 \end{array}$$

Solution: let A be the coefficient matrix. If $\det(A) \neq 0$, then A is invertible and the system will have a unique solution. The other two cases can happen only if $\det(A) = 0$.

$$\det \begin{bmatrix} 1 & -c & c \\ 1 & 1 & -1 \\ c & -c & 1 \end{bmatrix} = 1 + c^2 - c^2 - c^2 - c + c = 1 - c^2,$$

so $\det(A) = 0 \Leftrightarrow c = \pm 1$. Thus *(ii)*, there is a unique solution if $c \neq \pm 1$.

(iii) If $c = 1$, then the system is

$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & x_3 & = & 4 \\ x_1 & + & x_2 & - & x_3 & = & 2 \\ x_1 & - & x_2 & + & x_3 & = & 4 \end{array}$$

which has infinitely many solutions, since the first and third equations are identical.

(i) If $c = -1$, then the system is

$$\begin{array}{rcrcrcl} x_1 & + & x_2 & - & x_3 & = & 4 \\ x_1 & + & x_2 & - & x_3 & = & 2 \\ -x_1 & + & x_2 & + & x_3 & = & 4 \end{array}$$

which is inconsistent, since the first minus the second equation gives $0 = 2$.

Alternate approach: reduce the augmented matrix and consider its rank, and the rank of the coefficient matrix, as in Chapter 1.

$$\left[\begin{array}{ccc|c} 1 & -c & c & 4 \\ 1 & 1 & -1 & 2 \\ c & -c & 1 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & -c-1 & c+1 & 2 \\ 1 & 1 & -1 & 2 \\ 0 & -2c & c+1 & 4-2c \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 0 & -c-1 & c+1 & 2 \\ 1 & 1 & -1 & 2 \\ 0 & -c+1 & 0 & 2-2c \end{array}\right]$$

The rank of the coefficient matrix will be 3 if and only if $c \neq \pm 1$; now proceed as above.

4. [10 marks] Solve the following system of differential equations for f_1 and f_2 as functions of x if

$$\begin{array}{rcrcr} f_1' & = & 2f_1 & + & 3f_2 \\ f_2' & = & 4f_1 & + & 3f_2 \end{array}$$

and $f_1(0) = -3$, $f_2(0) = 1$.

Solution: need the eigenvalues and eigenvectors of the coefficient matrix.

$$\det \begin{bmatrix} \lambda - 2 & -3 \\ -4 & \lambda - 3 \end{bmatrix} = \lambda^2 - 5\lambda - 6 = (\lambda - 5)(\lambda + 1),$$

so the eigenvalues are $\lambda_1 = 6$ and $\lambda_2 = -1$. Now find the eigenvectors:

$\lambda_1 = 6$:

$$\left[\begin{array}{rr|r} 4 & -3 & 0 \\ -4 & 3 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 4 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right];$$

take

$$X_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

$\lambda_2 = -1$:

$$\left[\begin{array}{rr|r} -3 & -3 & 0 \\ -4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right];$$

take

$$X_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

So the general solution is

$$F = c_1 X_1 e^{\lambda_1 x} + c_2 X_2 e^{\lambda_2 x}.$$

Use the initial conditions to find c_1 and c_2:

$$\begin{bmatrix} f_1(0) \\ f_2(0) \end{bmatrix} = c_1 \begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix}^{-1} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} -1 & -1 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 2 \\ 15 \end{bmatrix}$$

Thus

$$f_1(x) = -\frac{6}{7} e^{6x} - \frac{15}{7} e^{-x}$$

and

$$f_2(x) = -\frac{8}{7} e^{6x} + \frac{15}{7} e^{-x}.$$

5. [10 marks] Consider triangle ΔPQR with vertices $P(1, 1, -1)$, $Q(1, 3, 2)$, $R(2, 3, 5)$. Find:

(a) [4 marks] the length of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$

Solution:

$$\overrightarrow{PQ} = \begin{bmatrix} 1-1 \\ 3-1 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}; \; \|\overrightarrow{PQ}\| = \sqrt{0^2+2^2+3^2} = \sqrt{13}$$

$$\overrightarrow{PR} = \begin{bmatrix} 2-1 \\ 3-1 \\ 5-(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}; \; \|\overrightarrow{PR}\| = \sqrt{1^2+2^2+6^2} = \sqrt{41}$$

(b) [3 marks] the interior angle at P, to the nearest degree

Solution:

$$\cos(\angle P) = \frac{\overrightarrow{PQ} \cdot \overrightarrow{PR}}{\|\overrightarrow{PQ}\|\|\overrightarrow{PR}\|} = \frac{0+4+18}{\sqrt{13}\sqrt{41}} = \frac{22}{\sqrt{13}\sqrt{41}} \Rightarrow \angle P \simeq 18^\circ$$

(c) [3 marks] the area of ΔPQR

Solution:

$$\text{area} = \frac{1}{2}\|\overrightarrow{PQ}\times\overrightarrow{PR}\| = \frac{1}{2}\left\|\begin{bmatrix} 12-6 \\ -(0-3) \\ 0-2 \end{bmatrix}\right\| = \frac{1}{2}\left\|\begin{bmatrix} 6 \\ 3 \\ -2 \end{bmatrix}\right\| = \frac{1}{2}\sqrt{36+9+4} = \frac{7}{2}$$

6. [7 marks] If

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \text{ and } \vec{d} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix},$$

express $\vec{v}$ in the form $\vec{v} = \vec{v}_1 + \vec{v}_2$ where $\vec{v}_1$ is parallel to $\vec{d}$ and $\vec{v}_2$ is orthogonal to $\vec{d}$.

Solution: use the projection formula.

$$\vec{v}_1 = \text{proj}_{\vec{d}}\vec{v} = \frac{\vec{v} \cdot \vec{d}}{\vec{d} \cdot \vec{d}}\vec{d} = \frac{3-2-3}{9+1+1} = -\frac{2}{11}\vec{d} = -\frac{2}{11}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

Then

$$\vec{v}_2 = \vec{v} - \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + \frac{2}{11}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 17 \\ 20 \\ -31 \end{bmatrix}.$$

7. [10 marks] Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 4 & 0 & 2 \end{bmatrix}$ and determine if A is diagonalizable.

Solution:

$$\begin{aligned} & \det(\lambda I - A) \\ = & \det \begin{bmatrix} \lambda - 2 & 0 & -4 \\ 0 & \lambda - 3 & 0 \\ -4 & 0 & \lambda - 2 \end{bmatrix} = (\lambda - 3)\det \begin{bmatrix} \lambda - 2 & -4 \\ -4 & \lambda - 2 \end{bmatrix} \\ = & (\lambda - 3)(\lambda^2 - 4\lambda - 12) \\ = & (\lambda - 3)(\lambda - 6)(\lambda + 2) \end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 3,\ \lambda_2 = 6$ and $\lambda_3 = -2$. A is diagonalizable since A has three distinct eigenvalues.

Eigenvectors:

$$(3I - A|O) = \left[\begin{array}{rrr|r} 1 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ -4 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];\ \text{take } X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$(6I - A|O) = \left[\begin{array}{rrr|r} 4 & 0 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ -4 & 0 & 4 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];\ \text{take } X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

$$(-2I - A|O) = \left[\begin{array}{rrr|r} -4 & 0 & -4 & 0 \\ 0 & -5 & 0 & 0 \\ -4 & 0 & -4 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right];\ \text{take } X_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Note: you could also say A is diagonalizable since $P = \left[\ X_1 \mid X_2 \mid X_3\ \right]$ is invertible, because $\det(P) = -2 \neq 0$.

8. [9 marks; 3 marks for each part] Indicate if the following statements are True or False, and give a brief explanation why.

(a) Every invertible matrix is diagonalizable. **True** **False**

Soluiton: False. Consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

It has eigenvalue $\lambda = 1$, repeated. So if it were diagonalizable, there would be an invertible matrix P such that

$$D = P^{-1}AP \Leftrightarrow I = P^{-1}AP \Leftrightarrow PIP^{-1} = A \Leftrightarrow I = A,$$

which is not true. So A is not diagonalizable. But it is invertible, since

$$\det(A) = 1 \neq 0.$$

(b) $\vec{u} \cdot (\vec{v} \times \vec{w}) = -\vec{v} \cdot (\vec{u} \times \vec{w})$ **True** **False**

Solution: True.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \det \left[\begin{array}{c|c|c} \vec{u} & \vec{v} & \vec{w} \end{array} \right] = -\det \left[\begin{array}{c|c|c} \vec{v} & \vec{u} & \vec{w} \end{array} \right] = -\vec{v} \cdot (\vec{u} \times \vec{w})$$

(c) If A is a $n \times n$ matrix and $A^T = A^{-1}$, then $\det(A) = \pm 1$. **True** **False**

Solution: True.

$$\begin{aligned} A^T = A^{-1} \quad &\Rightarrow \quad A^T A = I \\ &\Rightarrow \quad \det(A^T A) = \det I \\ &\Rightarrow \quad \det(A^T) \det A = 1 \\ &\Rightarrow \quad (\det(A))^2 = 1 \\ &\Rightarrow \quad \det(A) = \pm 1 \end{aligned}$$