University of Toronto
Solutions to **MAT188H1F TERM TEST**
of **Tuesday, November 10, 2009**
Duration: 90 minutes

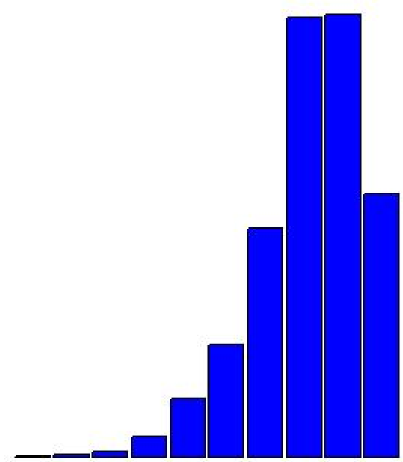
General Comments about the Test:

1. Regarding True or False questions: you can't prove a statement is true by giving an example; you must give a general argument. On the other hand, to prove a statement is false it is sufficient to give one counter example.

2. If you use a calculator to give an answer in decimal form then you are approximating the answer, which will cost you marks. For example, in Question #2 the distance from the point to the plane is $\sqrt{2}/3$. Any decimal value you give for this will necessarily be an approximation since $\sqrt{2}/3$ is an irrational number with a never-ending, never-repeating decimal expansion. If it doesn't ask for an approximation then don't approximate it!

Breakdown of Results: 905 students wrote this test. The marks ranged from 1.7% to 100%, and the average was 75.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	16.7%
A	44.8%	80-89%	28.1%
B	27.8%	70-79%	27.8%
C	14.5%	60-69%	14.5%
D	7.2%	50-59%	7.2%
F	5.7%	40-49%	3.8%
		30-39%	1.3%
		20-29%	0.3%
		10-19%	0.2%
		0-9%	0.1%



1. [8 marks] Consider the three points in space: $A(2,0,-1), B(5,1,2), C(4,2,1)$.

   (a) [5 marks] Find the scalar equation of the plane passing through the three points A, B and C.

   Solution: Let $\vec{u} = \overrightarrow{AB} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$ and $\vec{v} = \overrightarrow{AC} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$. Then a normal vector to the plane is given by

$$\begin{aligned} \vec{n} = \vec{u} \times \vec{v} &= \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \times \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}; \end{aligned}$$

   and the scalar equation of the plane is

$$\vec{n} \cdot \begin{bmatrix} x-2 \\ y-0 \\ z+1 \end{bmatrix} = 0 \Leftrightarrow -4x + 4z = -8 - 4 \Leftrightarrow x - z = 3.$$

   Alternates: You could just as well take

$$\vec{n} = \overrightarrow{BA} \times \overrightarrow{BC} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

   or

$$\vec{n} = \overrightarrow{CB} \times \overrightarrow{CA} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}$$

   (b) [3 marks] What is the area of the triangle with vertices A, B and C?

   Solution: area $= \dfrac{1}{2}\|\vec{u} \times \vec{v}\| = \dfrac{1}{2}\sqrt{(-4)^2 + 4^2} = 2\sqrt{2}$.

2. [8 marks] Find the minimum distance from the plane with equation $x - y + 4z = 6$ to the point $P(1, 1, 1)$.

Solution: the normal to the plane is $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ and $P_0(6, 0, 0)$ is a point on the plane. Then the minimum distance from the plane to the point P is given by

$$\|\text{proj}_{\vec{n}}\overrightarrow{P_0P}\| = \frac{|\vec{n} \cdot \overrightarrow{P_0P}|}{\|\vec{n}\|} = \frac{|(1)(-5) + (-1)(1) + (4)(1)|}{\sqrt{1 + 1 + 16}} = \frac{2}{\sqrt{18}} = \frac{\sqrt{2}}{3}$$

Alternate Solution: the line passing through P and normal to the plane has parametric equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}.$$

The point of intersection, Q, of this line and the plane is the point on the plane closest to P. To find the coordinates of Q, substitute from the line into the plane and solve for t :

$$1 + t - (1 - t) + 4(1 + 4t) = 6 \Leftrightarrow t = \frac{1}{9}.$$

Q has coordinates

$$(x, y, z) = \left(1 + \frac{1}{9}, 1 - \frac{1}{9}, 1 + \frac{4}{9}\right) = \left(\frac{10}{9}, \frac{8}{9}, \frac{13}{9}\right).$$

Then the minimum distance from the plane to the point P is given by

$$\|\overrightarrow{QP}\| = \sqrt{\left(-\frac{1}{9}\right)^2 + \left(\frac{1}{9}\right)^2 + \left(-\frac{4}{9}\right)^2} = \frac{\sqrt{18}}{9} = \frac{\sqrt{2}}{3},$$

as before.

3. [9 marks] The parts of this question are unrelated.

(a) [5 marks] Use Cramer's rule to solve for y if $\left\{\begin{array}{rcrcrcr} 5x & + & 3y & - & 2z & = & 4 \\ 3x & - & 2y & - & 2z & = & 3 \\ 2x & - & 4y & - & 3z & = & 3 \end{array}\right.$

Solution: you must use Cramer's Rule. No marks for any other method.

$$y = \frac{\det \begin{bmatrix} 5 & 4 & -2 \\ 3 & 3 & -2 \\ 2 & 3 & -3 \end{bmatrix}}{\det \begin{bmatrix} 5 & 3 & -2 \\ 3 & -2 & -2 \\ 2 & -4 & -3 \end{bmatrix}} = -\frac{1}{21}$$

since

$$\det \begin{bmatrix} 5 & 4 & -2 \\ 3 & 3 & -2 \\ 2 & 3 & -3 \end{bmatrix} = \det \begin{bmatrix} 5 & 2 & -2 \\ 3 & 1 & -2 \\ 2 & 0 & -3 \end{bmatrix} = \det \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -2 \\ 2 & 0 & -3 \end{bmatrix} = -1$$

and

$$\det \begin{bmatrix} 5 & 3 & -2 \\ 3 & -2 & -2 \\ 2 & -4 & -3 \end{bmatrix} = \det \begin{bmatrix} 5 & 1 & -2 \\ 3 & -4 & -2 \\ 2 & -7 & -3 \end{bmatrix} = \det \begin{bmatrix} 5 & 1 & -2 \\ 23 & 0 & -10 \\ 37 & 0 & -17 \end{bmatrix} = 21$$

(b) [4 marks] Find the adjoint of the inverse of A, if $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

Solution: this is a very short question, if you look at it the right way:

$$\begin{aligned} A^{-1} \operatorname{adj}(A^{-1}) = \det(A^{-1})\, I \quad &\Rightarrow \quad \operatorname{adj}(A^{-1}) = \det(A^{-1})\, A \\ &\Rightarrow \quad \operatorname{adj}(A^{-1}) = (\det(A))^{-1}\, A \\ &\Rightarrow \quad \operatorname{adj}(A^{-1}) = A \end{aligned}$$

since

$$\det \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \det \begin{bmatrix} 0 & -3 & -1 \\ 0 & 4 & 1 \\ 1 & 1 & 2 \end{bmatrix} = -3 + 4 = 1.$$

4. [10 marks] Find both $f_1(x)$ and $f_2(x)$ if $\left\{ \begin{array}{rcrcr} f_1' & = & f_1 & + & 8f_2 \\ f_2' & = & f_1 & + & 3f_2 \end{array} \right.$ and $f_1(0) = f_2(0) = 4$.

Solution: let $A = \begin{bmatrix} 1 & 8 \\ 1 & 3 \end{bmatrix}$. Then

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1 & -8 \\ -1 & \lambda - 3 \end{bmatrix} = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$

Eigenvalues of $A : (\lambda - 5)(\lambda + 1) = 0 \Leftrightarrow \lambda = 5$ or $\lambda = -1$.

Eigenvectors of A :

$$(5I - A|0) \rightarrow \left[\begin{array}{rr|r} 4 & -8 & 0 \\ -1 & 2 & 0 \end{array} \right] \rightarrow \left[\begin{array}{rr|r} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right]; \text{ take } X_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

and

$$(-I - A|0) \rightarrow \left[\begin{array}{rr|r} -2 & -8 & 0 \\ -1 & -4 & 0 \end{array} \right] \rightarrow \left[\begin{array}{rr|r} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array} \right]; \text{ take } X_2 = \begin{bmatrix} -4 \\ 1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5x} + c_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} e^{-x}.$$

To find c_1 and c_2, let $x = 0$:

$$\begin{bmatrix} 4 \\ 4 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -4 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Leftrightarrow c_1 = \frac{10}{3}, \quad c_2 = \frac{2}{3}.$$

Thus,

$$f_1(x) = \frac{20}{3} e^{5x} - \frac{8}{3} e^{-x}$$

and

$$f_2(x) = \frac{10}{3} e^{5x} + \frac{2}{3} e^{-x}.$$

5. [8 marks] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ x-3y \end{bmatrix}.$$

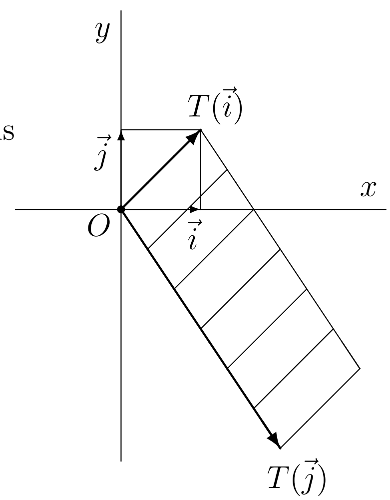
(a) [4 marks] Sketch the image of the unit square.

Solution:

$$T(\vec{i}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } T(\vec{j}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

So the image of the unit square is the parallelogram determined by

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$



(b) [4 marks] Find the formula for $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

Solution: The matrix of T is

$$A = [T(\vec{i})|T(\vec{j})] = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}.$$

Then the matrix of T^{-1} is

$$A^{-1} = -\frac{1}{5}\begin{bmatrix} -3 & -2 \\ -1 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix},$$

and the formula for T^{-1} is

$$\begin{aligned} T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix} 3x+2y \\ x-y \end{bmatrix} \end{aligned}$$

6. [8 marks] The parts of this question are unrelated.

(a) [4 marks] Find the matrix that represents a reflection in the line $y = -4x$.

Solution: recall the formula from the book. A reflection in the line $y = mx$ has matrix

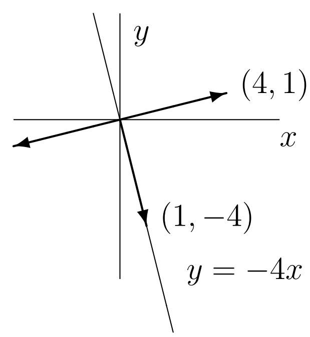
$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

So with $m = -4$, the matrix is

$$\frac{1}{17}\begin{bmatrix} -15 & -8 \\ -8 & 15 \end{bmatrix}.$$

Alternate Solution: let the matrix be Q.
Geometrically,

$$Q\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \text{ and } Q\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -1 \end{bmatrix},$$



since $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ is parallel to the line $y = -4x$,
and $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is perpendicular to it.
Then

$$Q\begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ -4 & -1 \end{bmatrix} \Leftrightarrow Q = \begin{bmatrix} 1 & -4 \\ -4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}^{-1}$$

$$\Leftrightarrow Q = \begin{bmatrix} 1 & -4 \\ -4 & -1 \end{bmatrix}\frac{1}{17}\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$$

$$\Leftrightarrow Q = \frac{1}{17}\begin{bmatrix} -15 & -8 \\ -8 & 15 \end{bmatrix}$$

(b) [4 marks] Find R^{45} if $R = \begin{bmatrix} \cos(\pi/9) & -\sin(\pi/9) \\ \sin(\pi/9) & \cos(\pi/9) \end{bmatrix}$. (Hint: what does R represent geometrically?)

Solution: R is a rotation of $\pi/9$ radians counter clockwise around the origin. So R^{45} is a rotation of

$$45 \times \frac{\pi}{9} = 5\pi$$

radians counter clockwise around the origin. Hence

$$R^{45} = \begin{bmatrix} \cos(5\pi) & -\sin(5\pi) \\ \sin(5\pi) & \cos(5\pi) \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

7. [9 marks; 3 marks for each part] Indicate if the following statements are True or False, and give a brief explanation why.

(a) If A is a 5×5 matrix such that $A^T = -A$, then $\det(A) = 0$. **True** **False**

Solution: True.

$$\begin{aligned} A^T = -A \quad &\Rightarrow \quad \det(A^T) = \det(-A) \\ &\Rightarrow \quad \det A = (-1)^5 \det A \\ &\Rightarrow \quad \det A = -\det A \\ &\Rightarrow \quad \det A = 0 \end{aligned}$$

(b) If 3 is an eigenvalue of A, then 9 is an eigenvalue of A^2. **True** **False**

Solution: True. Let X be an eigenvector of A corresponding to the eigenvalue 3. Then

$$\begin{aligned} AX = 3X \quad &\Rightarrow \quad A(AX) = A(3X) \\ &\Rightarrow \quad A^2X = 3AX \\ &\Rightarrow \quad A^2X = 3(3X) \\ &\Rightarrow \quad A^2X = 9X \end{aligned}$$

(c) If $\vec{d} \neq \vec{0}$, then $\|\text{proj}_{\vec{d}}(\vec{v})\| = \|\vec{v}\| \cos\theta$, where θ is the angle between $\vec{d}$ and $\vec{v}$.
True **False**

Solution: False. It's only true if $\cos\theta \geq 0$.

$$\begin{aligned} \|\text{proj}_{\vec{d}}(\vec{v})\| \quad &= \quad \left\| \frac{\vec{v} \cdot \vec{d}}{\|\vec{d}\|^2} \vec{d} \right\| \\ &= \quad \frac{|\vec{v} \cdot \vec{d}|}{\|\vec{d}\|} \\ &= \quad \frac{\|\vec{v}\| \, \|\vec{d}\| | \cos\theta|}{\|\vec{d}\|} \\ &= \quad \|\vec{v}\| \, |\cos\theta| \end{aligned}$$

A counterexample: take $\vec{d} = \vec{i}$ and $\vec{v} = -\vec{i}$. Then $\theta = \pi$ and $\|\vec{v}\| \cos\theta = -1$ but $\|\text{proj}_{\vec{d}}(\vec{v})\| = \| - \vec{i}\| = 1$.