University of Toronto
Solutions to **MAT188H1F TERM TEST**
of **Tuesday, November 11, 2008**
Duration: 90 minutes

General Comments about the Test:

1. Questions 1, 2, 4 and 5 are based directly on homework problems; these questions should all have been aced.

2. Question 3, the subspace question, could have been done by using the definition of a subspace, but it is easier to realize that the given set U is just the null space of the matrix $A - A^T$. Reducing the given matrix A in Question 3 has absolutely nothing to do with the solution and is just a waste of time.

3. Question 3 is actually a slight variation of homework problem #18 from Section 4.1

4. Questions 4 and 6 were the only two questions requiring you to find eigenvalues and eigenvectors; both involved only 2×2 matrices. Finding the eigenvalues and eigenvectors should have been no problem at all.

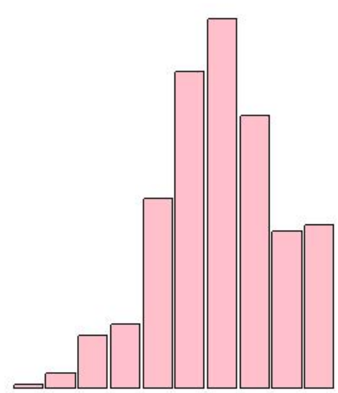
5. Question 7(c) is homework problem #16(f) of Section 3.2, which the answers in the back of the book list as True – even though Nicholson's 'definition' of orthogonal vectors on page 145 says

   > The nonzero vectors $\vec{v}$ and $\vec{w}$ are called **orthogonal** if the angle between them θ is a right angle, that is, if $\theta = \frac{\pi}{2}$.

   So if you said 7(c) was False *because* $\vec{v} = \vec{0}$ *is not orthogonal to* $\vec{d}$*, according to Nicholson's definition,* you will get full marks. (Note though that his definition doesn't include the case when $\vec{u}$ or $\vec{v}$ are zero. The zero vector *is* considered orthogonal to every vector.)

Breakdown of Results: 847 students wrote this test. The marks ranged from 6.7% to 100%, and the average was 63.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	10.2%
A	20.0%	80-89%	9.8%
B	17.0%	70-79%	17.0%
C	23.0%	60-69%	23.0%
D	19.7%	50-59%	19.7%
F	20.3%	40-49%	11.8%
		30-39%	4.0%
		20-29%	3.3%
		10-19%	1.0 %
		0-9%	0.2%



1. [6 marks] Find the scalar equation of the plane that passes through the three points

$$A(1, 1, -1), B(3, 3, 2), C(5, 0, -4).$$

Solution: Let $\vec{u} = \overrightarrow{AB} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \overrightarrow{AC} = \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then a normal vector to the plane is given by

$$\begin{aligned} \vec{n} = \vec{u} \times \vec{v} &= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ -1 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 18 \\ -10 \end{bmatrix}; \end{aligned}$$

and the scalar equation of the plane is

$$\vec{n} \cdot \vec{x} = \vec{n} \cdot \overrightarrow{OA} \Leftrightarrow -3x + 18y - 10z = -3 + 18 + 10 \Leftrightarrow 3x - 18y + 10z = -25.$$

NOTE: instead of $\overrightarrow{OA}$ you could just as well have used $\overrightarrow{OB}$ or $\overrightarrow{OC}$ and gotten the same equation.

Alternates: You could just as well take

$$\vec{n} = \overrightarrow{BA} \times \overrightarrow{BC} = \begin{bmatrix} 3 \\ -18 \\ 10 \end{bmatrix}$$

or

$$\vec{n} = \overrightarrow{CB} \times \overrightarrow{CA} = \begin{bmatrix} 3 \\ -18 \\ 10 \end{bmatrix}$$

2. [8 marks] Find the solution to the system of linear equations

$$\begin{cases} x + 2y + 3z = -1 \\ -x + 2y + z = 5 \\ -x + 4y + 3z = 7 \end{cases}$$

that is closest to the point $P(1,1,1)$.

Solution: solve the system by reducing the augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ -1 & 2 & 1 & 5 \\ -1 & 4 & 3 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 4 & 4 & 4 \\ 0 & 6 & 6 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So the solution to the above linear system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

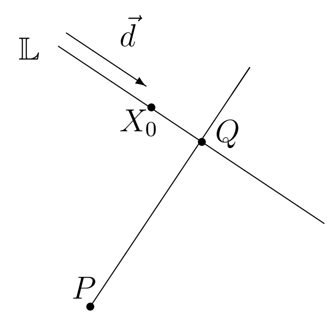
which is the vector equation of a line; call it $\mathbb{L}$. Geometrically, the problem is: find the point on $\mathbb{L}$ closest to the point P. Let this point be Q with coordinates (a,b,c).

The line $\mathbb{L}$ has direction vector

$$\vec{d} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

and passes through the point

$$X_0(-3,1,0).$$



From the diagram, $\overrightarrow{PQ} + \text{proj}_{\vec{d}}\overrightarrow{PX_0} = \overrightarrow{PX_0} \Leftrightarrow \overrightarrow{PQ} = \overrightarrow{PX_0} - \text{proj}_{\vec{d}}\overrightarrow{PX_0}$. Hence

$$\begin{bmatrix} a-1 \\ b-1 \\ c-1 \end{bmatrix} = \begin{bmatrix} -3-1 \\ 1-1 \\ 0-1 \end{bmatrix} - \frac{1}{3}\left(\begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right)\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

So the point $Q(-2,2,-1)$ is the point on the line of intersection of the above three planes that is closest to the point $P(1,1,1)$.

3. [10 marks] Let $A = \begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 3 & -2 & 1 \\ 3 & -2 & 5 & 6 \\ -1 & 1 & 7 & 4 \end{bmatrix}$; let $U = \{X \in \mathbb{R}^4 | AX = A^T X\}$.

(a) [5 marks] Show that U is a subspace of $\mathbb{R}^4$.

Solution: Short way: $AX = A^T X \Leftrightarrow (A - A^T)X = O$; so $U = \text{null}(A - A^T)$, which is a subspace. Alternately, you could use the definition of subspace:

1. U is non-empty: $AO = O = A^T O \Rightarrow O \in U$.
2. U is closed under addition:

$$\begin{aligned} X \in U, Y \in U &\Rightarrow AX = A^T X \text{ and } AY = A^T Y \\ &\Rightarrow A(X+Y) = AX + AY = A^T X + A^T Y = A^T(X+Y) \\ &\Rightarrow X + Y \in U \end{aligned}$$

3. U is closed under scalar multiplication:

$$\begin{aligned} X \in U, a \in \mathbb{R} &\Rightarrow AX = A^T X \\ &\Rightarrow A(aX) = aAX = aA^T X = A^T(aX) \\ &\Rightarrow aX \in U \end{aligned}$$

(b) [5 marks] Find a spanning set for U.

$$\begin{aligned} \text{null}(A - A^T) &= \text{null}\left(\begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 3 & -2 & 1 \\ 3 & -2 & 5 & 6 \\ -1 & 1 & 7 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 3 & -1 \\ 0 & 3 & -2 & 1 \\ 4 & -2 & 5 & 7 \\ -1 & 1 & 6 & 4 \end{bmatrix} \right) \\ &= \text{null} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \end{aligned}$$

4. [10 marks] Find all the solutions, $f_1(x), f_2(x)$, if

$$\left\{ \begin{array}{rcrcr} f_1' & = & -f_1 & + & 5f_2 \\ f_2' & = & f_1 & + & 3f_2 \end{array} \right.$$

Solution: Let $A = \begin{bmatrix} -1 & 5 \\ 1 & 3 \end{bmatrix}$; then

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda + 1 & -5 \\ -1 & \lambda - 3 \end{bmatrix} = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2).$$

So the eigenvalues of A are $\lambda_1 = 4$ and $\lambda_2 = -2$.

Find eigenvectors. (NOTE: I am using the notation from Example 7, Section 4.1)

$$E_4(A) = \text{null}\,(4I - A) = \text{null} \begin{bmatrix} 5 & -5 \\ -1 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

and

$$E_{-2}(A) = \text{null}\,(-2I - A) = \text{null} \begin{bmatrix} -1 & -5 \\ -1 & -5 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -5 \\ 1 \end{bmatrix} \right\}.$$

Thus

$$\begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4x} + c_2 \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-2x};$$

that is,

$$f_1(x) = c_1 e^{4x} - 5c_2 e^{-2x}$$

and

$$f_2(x) = c_1 e^{4x} + c_2 e^{-2x}$$

for arbitrary constants c_1, c_2.

5. [7 marks] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ -x+3y \end{bmatrix}.$$

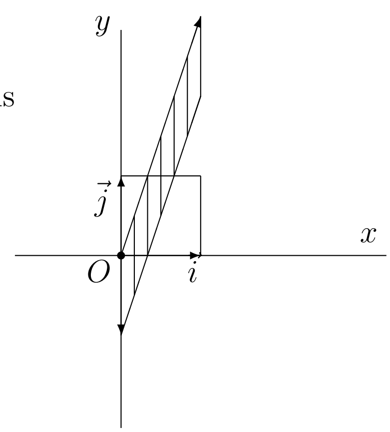
(a) [3 marks] Sketch the image of the unit square.

Solution:

$$T(\vec{i}) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ and } T(\vec{j}) = T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

So the image of the unit square is the parallelogram determined by

$$\begin{bmatrix} 0 \\ -1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$



(b) [4 marks] Find the formula for $T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

Solution: The standard matrix of T is

$$A = [T(\vec{i})|T(\vec{j})] = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}.$$

Then the standard matrix of T^{-1} is

$$A^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix},$$

and the formula for T^{-1} is

$$\begin{aligned} T^{-1}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 3x-y \\ x \end{bmatrix} \end{aligned}$$

6. [10 marks] Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a reflection in the line $y = 3x$.

(a) [2 marks] Find the matrix of T.

Solution: Recall the formula from the book: a reflection in the line $y = 3x$ has matrix

$$A = \frac{1}{1+3^2}\begin{bmatrix} 1-3^2 & 2(3) \\ 2(3) & 3^2-1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -8 & 6 \\ 6 & 8 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -4 & 3 \\ 3 & 4 \end{bmatrix}.$$

(b) [6 marks] Find the eigenvalues and eigenvectors of the matrix of T.

Solution:

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda + 4/5 & -3/5 \\ -3/5 & \lambda - 4/5 \end{bmatrix} = \lambda^2 - 16/25 - 9/25 = \lambda^2 - 1.$$

So the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. Find eigenvectors. (NOTE: I am using the notation from Example 7, Section 4.1)

$$E_1(A) = \text{null}\,(I - A) = \text{null}\begin{bmatrix} 9/5 & -3/5 \\ -3/5 & 1/5 \end{bmatrix} = \text{null}\begin{bmatrix} 3 & -1 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\};$$

$$E_{-1}(A) = \text{null}\,(-I - A) = \text{null}\begin{bmatrix} -1/5 & -3/5 \\ -3/5 & -9/5 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}.$$

So

$$\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}; \text{ and } \lambda_2 = -1, \vec{v}_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

(c) [2 marks] Interpret your results from part (b) geometrically.

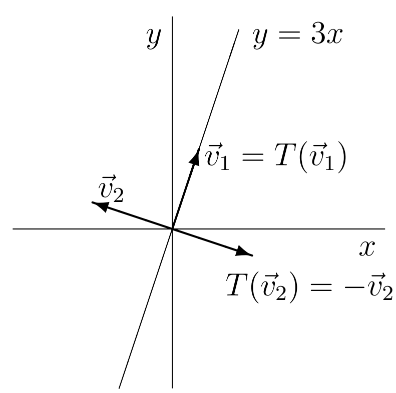
Solution: Observe that $\vec{v}_1$ is parallel to the line with equation $y = 3x$, and $\vec{v}_2$ is orthogonal to the line, since $\vec{v}_1 \cdot \vec{v}_2 = 0$.

1. $\vec{v}_1$ is on the axis of reflection, so

$$T(\vec{v}_1) = \vec{v}_1 \Leftrightarrow A\vec{v}_1 = \vec{v}_1.$$

2. $\vec{v}_2$ is orthogonal to the axis of reflection, so

$$T(\vec{v}_2) = -\vec{v}_2 \Leftrightarrow A\vec{v}_2 = -\vec{v}_2.$$



7. [9 marks; 3 marks for each part] Indicate if the following statements are True or False, and give a brief explanation why.

(a) If $\{X, Y, Z\}$ is a linearly independent set in $\mathbb{R}^n$, then $\{X+Y, Y+Z, Z+X\}$ is also a linearly independent set in $\mathbb{R}^n$. **True** **False**

True:

$$\begin{aligned}
& a(X+Y)+b(Y+Z)+c(Z+X)=O \\
\Rightarrow\ & (a+c)X+(a+b)Y+(b+c)Z=O \\
\Rightarrow\ & \left\{\begin{array}{rcrcrcl} a & & & + & c & = & 0 \\ a & + & b & & & = & 0 \\ & & b & + & c & = & 0 \end{array}\right. , \text{ since } X, Y, Z \text{ are independent} \\
\Rightarrow\ & \left\{\begin{array}{rcrcrcl} a & & & + & c & = & 0 \\ & & b & - & c & = & 0 \\ & & b & + & c & = & 0 \end{array}\right. \\
\Rightarrow\ & \left\{\begin{array}{rcrcrcl} a & & & + & c & = & 0 \\ & & b & - & c & = & 0 \\ & & & & 2c & = & 0 \end{array}\right. \\
\Rightarrow\ & c=0, b=0, a=0
\end{aligned}$$

(b) Every plane containing the points $A(1,-3,2)$ and $B(4,5,0)$ also contains the point $C(7,13,-2)$. **True** **False**

True: every plane containing the points A and B will contain the line joining A and B. It turns out that C is on that line:

$$\begin{aligned}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 7 \\ 13 \\ -2 \end{bmatrix} \Leftrightarrow t = 1
\end{aligned}$$

SIMPLER: $\overrightarrow{AC}$ and $\overrightarrow{AB}$ are parallel, so A, B and C are collinear.

(c) If the projection of a vector $\vec{v}$ onto the non-zero vector $\vec{d}$ is the zero vector, then $\vec{v}$ is orthogonal to $\vec{d}$. **True** **False**

True:

$$\text{proj}_{\vec{d}}\vec{v} = \vec{0} \Leftrightarrow \frac{\vec{v} \cdot \vec{d}}{\|\vec{d}\|^2}\vec{d} = \vec{0} \Leftrightarrow \vec{v} \cdot \vec{d} = 0, \text{ since } \vec{d} \neq \vec{0}.$$