University of Toronto

Faculty of Applied Science and Engineering

Solutions to Final Examination, December 2019

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1F - Linear Algebra

Examiners: D. Burbulla, S. Cohen, L. Döppenschmitt, M. Greeff,
J. Sivaraman, S. Sorkhou, S. Uppal

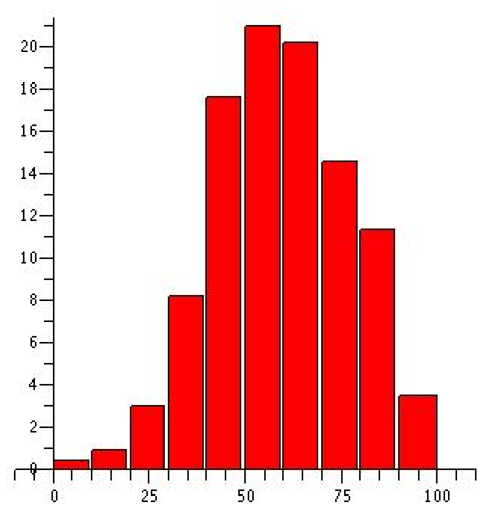
Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

- Many students lost marks for using notation incorrectly; e.g. in Question 1(a): $\text{col}(A) = \{\vec{c}_1, \vec{c}_2, \vec{c}_5\}$ is *wrong* since the left side consists of an infinite number of vectors, but the right side contains only three vectors.
-

Breakdown of Results: 911 registered students wrote this test. The marks ranged from 5% to 98.75% and the average was 47.438/80 or 59.3%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.8%
A	15.0%	80-89%	11.2%
B	14.4%	70-79%	14.4%
C	20.0%	60-69%	20.0%
D	20.7%	50-59%	20.7%
F	29.9%	40-49%	17.5 %
		30-39%	8.1%
		20-29%	3.0%
		10-19%	0.9%
		0-9%	0.4%



1. [avg: 7.33/10] The reduced row echelon form of

$$A = \begin{bmatrix} 2 & 1 & 1 & 1 & 6 \\ -1 & 2 & -8 & 7 & -7 \\ -2 & -1 & -1 & -1 & -6 \\ 1 & 1 & -1 & 2 & 1 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(a) [7 marks] Find a basis for each of col(A) and null(A).

Solution: a basis for col(A) consists of the columns of A that correspond to the columns of R with leading 1's:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -7 \\ -6 \\ 1 \end{bmatrix} \right\},$$

OR any three *independent* columns of A, which you must demonstrate *are* independent.

A basis for null(A) consists of the basic solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, which can be read off R :

$$\left\{ \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Aside: if not by inspection then you can get the basic solutions to $A\vec{x} = \vec{0}$ by finding the general solution and writing it as a linear combination, where s and t are parameters:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2s+t \\ 3s-3t \\ s \\ t \\ 0 \end{bmatrix} = s \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

(b) [3 marks] Is $\beta = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -8 \\ 7 \\ -7 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ -1 \\ -1 \\ -6 \end{bmatrix} \right\}$ a basis for row(A)? Explain why, or why not.

Solution: No. They are not linearly independent: the third vector is the negative of the first.

2.[avg: 6.9/10] Suppose $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{5}\begin{bmatrix} -3x+4y \\ 4x+3y \end{bmatrix}$.

(a) [7 marks] Let A be the matrix of T. Find the eigenvalues of A and a basis for each eigenspace of A.

Solution: $A = [T(\vec{e}_1)\ T(\vec{e}_2)] = \begin{bmatrix} -3/5 & 4/5 \\ 4/5 & 3/5 \end{bmatrix}$. Then

$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda + 3/5 & -4/5 \\ -4/5 & \lambda - 3/5 \end{bmatrix} = \lambda^2 - \frac{9}{25} - \frac{16}{25} = \lambda^2 - 1.$$

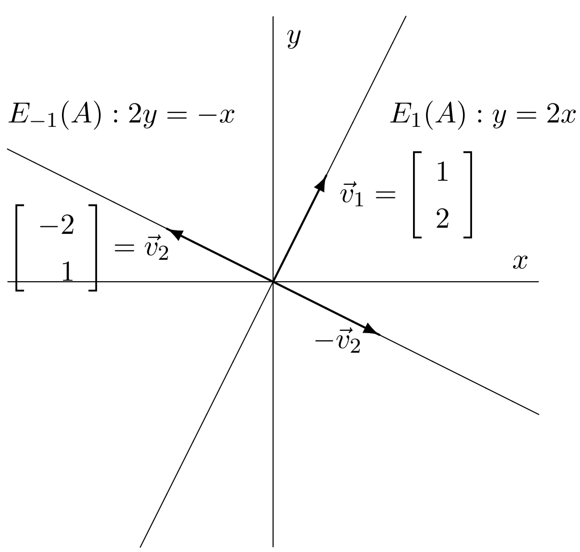
So the eigenvalues of A are $\lambda_1 = 1,\ \lambda_2 = -1$. Find a basis for each eigenspace:

$$E_1(A) = \text{null}\begin{bmatrix} 1+3/5 & -4/5 \\ -4/5 & 1-3/5 \end{bmatrix} = \text{null}\begin{bmatrix} 8 & -4 \\ -4 & 2 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}.$$

$$E_{-1}(A) = \text{null}\begin{bmatrix} -1+3/5 & -4/5 \\ -4/5 & -1-3/5 \end{bmatrix} = \text{null}\begin{bmatrix} -2 & -4 \\ -4 & -8 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}.$$

(b) [3 marks] Sketch the eigenspaces of A in the plane and interpret your results from part (a) geometrically in terms of the transformation T. You must clearly identify each eigenspace.

Solution: $E_1(A)$ is the line with equation $y = 2x$; $E_{-1}(A)$ is the line with equation $2y = -x$.



A is a reflection matrix with $m = 2$. So T is a reflection in the line with equation $y = 2x$. Since

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is parallel to the axis of reflection of T we have

$$T(\vec{v}_1) = \vec{v}_1.$$

Since

$$\vec{v}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

is orthogonal to the axis of reflection of T we have

$$T(\vec{v}_2) = -\vec{v}_2.$$

3. [avg: 5.1/10] For each of the following subsets U of $\mathbb{R}^3$ determine if it is a subspace of $\mathbb{R}^3$. If it is, find a basis for U and state its dimension.

(a) [4 marks] $U = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ in } \mathbb{R}^3 \mid xyz = 0 \right\}$.

Solution: No. U is not closed under vector addition:

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ are both in } U, \text{ but } \vec{u} + \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ isn't.}$$

Aside: U does contain the zero vector and *is* closed under scalar multiplication.

(b) [6 marks] $U = \{\vec{x} \text{ in } \mathbb{R}^3 \mid \vec{x} \cdot \vec{u}_1 = 0 \text{ and } \vec{x} \cdot \vec{u}_2 = 0\}$, if $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \ \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: Yes. By definition U is the orthogonal complement of the subspace $W = \operatorname{span}\{\vec{u}_1, \vec{u}_2\}$. Then

$$\dim(U) = 3 - \dim(W) = 3 - 2 = 1,$$

and a basis for U is the vector

$$\vec{u}_1 \times \vec{u}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}.$$

4. [10 marks; avg: 8.3/10]

4.(a) [5 marks] Find the least squares approximating line $y = a + b\,x$ to the four data points $(x, y) = (-1,-6), (0,-1), (1, 2), (2, 3)$.

Solution: use the normal equations. Let

$$M = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix},\ \vec{z} = \begin{bmatrix} a \\ b \end{bmatrix},\ Y = \begin{bmatrix} -6 \\ -1 \\ 2 \\ 3 \end{bmatrix};\ \text{then } M^T = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix}.$$

Solve the normal equations for $\vec{z}$:

$$M^T M \vec{z} = M^T Y \Leftrightarrow \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2 \\ 14 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 14 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 14 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} -40 \\ 60 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

So the least squares approximating line to the data has equation $y = -2 + 3\,x$.

4.(b) [5 marks] Find an orthogonal basis of span $\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ -7 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 2 \\ 0 \\ -5 \end{bmatrix} \right\}$.

Solution: use the Gram-Schmidt algorithm. Call the three given vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$, respectively. $\boxed{\text{Take } \vec{f}_1 = \vec{x}_1.}$ Then

$$\vec{f}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{f}_1}{\|\vec{f}_1\|^2} \vec{f}_1 = \begin{bmatrix} 3 \\ -5 \\ -7 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -4 \\ 1 \end{bmatrix},$$

$$\vec{f}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{f}_1}{\|\vec{f}_1\|^2} \vec{f}_1 - \frac{\vec{x}_3 \cdot \vec{f}_2}{\|\vec{f}_2\|^2} \vec{f}_2 = \begin{bmatrix} 7 \\ 2 \\ 0 \\ -5 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -3 \\ -2 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -2 \\ -4 \end{bmatrix};\ \text{or take } \vec{f}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ -2 \end{bmatrix}.$$

The orthogonal basis is $\{\vec{f}_1, \vec{f}_2, \vec{f}_3\}$. **Aside:** the orthogonal complement of span$\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ has basis $\vec{f}_4 = [1\ \ 4\ -2\ \ 3]^T$. Any three orthogonal vectors orthogonal to $\vec{f}_4$ would do.

5. [10 marks; 2 for each part. Avg: 4.4/10] Prove that the following statements are true. (Be concise.)

(a) If A is a 5×7 matrix and $\dim(\text{null}(A)) = 2$, then the matrix equation $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^5$.

Proof: $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in $\text{im}(A)$. So it suffices to show $\text{im}(A) = \mathbb{R}^5$:

$$\dim(\text{im}(A)) = 7 - \dim(\text{null}(A)) = 7 - 2 = 5.$$

That is, $\text{im}(A) = \mathbb{R}^5$.

(b) If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ are k linearly independent vectors in $\mathbb{R}^n$ and A is an invertible $n \times n$ matrix, then the vectors $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k$ are also linearly independent.

Proof: recall the columns of a matrix B are linearly independent if and only if $\text{null}(B) = \{0\}$. Let V be the $n \times k$ matrix $[\vec{v}_1\ \vec{v}_2\ \cdots\ \vec{v}_k]$. Let $\vec{x}$ be in $\text{null}(AV)$. Then

$$(AV)\vec{x} = \vec{0} \Rightarrow A(V\vec{x}) = \vec{0} \Rightarrow V\vec{x} = A^{-1}(\vec{0}) = \vec{0} \Rightarrow \vec{x} = 0.$$

Thus the columns of AV, namely $A\vec{v}_1, A\vec{v}_2, \ldots, A\vec{v}_k$, are linearly independent.

(c) If A is a $k \times n$ matrix such that $\text{null}(A) = \{\vec{0}\}$, then $n \le k$.

Proof: $k \ge \text{rank}(A) = n - \dim(\text{null}(A)) = n - 0 = n$.

(d) If A is a $k \times n$ matrix such that $\text{im}(A) = \mathbb{R}^k$, then $k \le n$.

Proof: $n \ge \text{rank}(A) = \dim(\text{im}(A)) = k$.

(e) If A is a 3×3 diagonalizable matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$, then

$$\det(A^2) = \lambda_1^2\lambda_2^2\lambda_3^2 \text{ and } \text{tr}(A^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

Proof: A is similar to the diagonal matrix $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$, and so A^2 is similar to the diagonal matrix $D^2 = \text{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2)$. In particular, this means

$$\det(A^2) = \det(D^2) = \lambda_1^2\lambda_2^2\lambda_3^2 \text{ and } \text{tr}(A^2) = \text{tr}(D^2) = \lambda_1^2 + \lambda_2^2 + \lambda_3^2.$$

6. [avg: 7.2/10]

Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if $A = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 7 & 12 \\ 0 & 12 & 0 \end{bmatrix}$.

Step 1: find the eigenvalues of A.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 16 & 0 & 0 \\ 0 & \lambda - 7 & -12 \\ 0 & -12 & \lambda \end{bmatrix} = (\lambda - 16) \det \begin{bmatrix} \lambda - 7 & -12 \\ -12 & \lambda \end{bmatrix}$$

$$= (\lambda - 16)(\lambda^2 - 7\lambda - 144) = (\lambda - 16)(\lambda - 16)(\lambda + 9) = (\lambda + 9)(\lambda - 16)^2$$

Thus the eigenvalues of A are $\lambda_1 = 16$, repeated, and $\lambda_2 = -9$.

Step 2: find an *orthogonal* basis of eigenvectors for each eigenspace.

$$E_{16}(A) = \text{null} \begin{bmatrix} 16 - 16 & 0 & 0 \\ 0 & 16 - 7 & -12 \\ 0 & -12 & 16 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 3 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix} \right\};$$

$$E_{-9}(A) = \text{null} \begin{bmatrix} -9 - 16 & 0 & 0 \\ 0 & -9 - 7 & -12 \\ 0 & -12 & -9 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 0 \\ -3 \\ 4 \end{bmatrix} \right\}.$$

Step 3: for the columns of P, take the unit, orthogonal eigenvectors and for the diagonal entries of D take the corresponding eigenvalues:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4/5 & -3/5 \\ 0 & 3/5 & 4/5 \end{bmatrix} \text{ and } D = \begin{bmatrix} 16 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & -9 \end{bmatrix},$$

7. [avg: 5.1/10] Let $\vec{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix}; \vec{x} = \begin{bmatrix} 3 \\ -4 \\ 5 \\ -4 \end{bmatrix}; S = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$. Let $U = \text{span}(S)$.

(a) [3 marks] Show that S is a basis for U.

Solution: since S is a spanning set for U, we need only show that S is linearly independent. The easy way to do this is to observe that S is actually an *orthogonal* set of non-zero vectors:

$$\vec{v}_1 \cdot \vec{v}_2 = 3 + 0 - 1 - 2 = 0;\ \vec{v}_1 \cdot \vec{v}_3 = 3 - 2 - 3 + 2 = 0;\ \vec{v}_2 \cdot \vec{v}_3 = 1 + 0 + 3 - 4 = 0.$$

As proved in class an orthogonal set of non-zero vectors is linearly independent. Thus S is a linearly independent, spanning set of U, and so S is a basis for U.

(b) [3 marks] Find a basis for $U^\perp$.

Solution:

$$U^\perp = \text{null}\begin{bmatrix} 3 & -1 & -1 & -1 \\ 1 & 0 & 1 & 2 \\ 1 & 2 & 3 & -2 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & -4 \\ 0 & 1 & 4 & 7 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

$$= \text{null}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix}\right\};\ \text{so } \vec{v}_4 = \begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix} \text{ is a basis for } U^\perp.$$

(c) [4 marks] Find vectors $\vec{u} \in U$ and $\vec{v} \in U^\perp$ such that $\vec{x} = \vec{u} + \vec{v}$.

Solution: by definition, $\vec{u} = \text{proj}_U(\vec{x})$ and $\vec{v} = \text{proj}_{U^\perp}(\vec{x})$. The easier one to calculate is $\vec{v}$, since $\dim(U^\perp) = 1$. Then take $\vec{u} = \vec{x} - \vec{v}$. The calculations are

$$\vec{v} = \text{proj}_{U^T}(\vec{x}) = \frac{\vec{x} \cdot \vec{v}_4}{\|\vec{v}_4\|^2}\vec{v}_4 = \frac{-36}{36}\begin{bmatrix} 1 \\ 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \\ 3 \\ -1 \end{bmatrix};\ \vec{u} = \vec{x} - \vec{v} = \begin{bmatrix} 3 \\ -4 \\ 5 \\ -4 \end{bmatrix} - \begin{bmatrix} -1 \\ -5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \\ -3 \end{bmatrix}.$$

Aside: for an alternate calculation of $\vec{u}$, see Page 10.

8. [avg: 3.0/10] Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x+y \\ 0 \end{bmatrix}$.

(a) [1 mark] Why is $U = \{T(\vec{v}) \mid \vec{v} \text{ in } \mathbb{R}^3\}$ a subspace of $\mathbb{R}^3$? (Short answer!)

Solution: $U = \text{im}(A)$, where $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ is the matrix of T.

(b) [3 marks] Find a basis for U. What is $\dim(U)$?

Solution: $\text{im}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right\}$. So $\dim(U) = 1$ and a basis for U is

$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

(c) [6 marks] Find the least possible value of $\|T(\vec{v}) - \vec{b}\|$, for $\vec{v}$ in $\mathbb{R}^3$, if

(*i*) $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ (*ii*) $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (*iii*) $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Solution: this is equivalent to asking what is the length of $\|\text{proj}_U(\vec{b}) - \vec{b}\|$, since $\text{proj}_U(\vec{b})$ is the vector in U closest to $\vec{b}$.

(*i*)	(*ii*)	(*iii*)
In this case $\vec{b} \in U$, so $\text{proj}_U(\vec{b}) = \vec{b}$. The least value of $\|T(\vec{v}) - \vec{b}\|$ is $\|\vec{b} - \vec{b}\| = \|\vec{0}\| = 0$.	In this case $\vec{b} \in U^\perp$, so $\text{proj}_U(\vec{b}) = \vec{0}$. The least value of $\|T(\vec{v}) - \vec{b}\|$ is $\|\vec{0} - \vec{b}\| = \|\vec{b}\| = 1$.	In this case $\text{proj}_U(\vec{b}) = \dfrac{\vec{b} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} = \dfrac{3}{2}\vec{u}$. The least value of $\|T(\vec{v}) - \vec{b}\|$ is $\left\|\dfrac{3}{2}\vec{u} - \vec{b}\right\| = \dfrac{\sqrt{38}}{2}$.

Some Alternate Calculations and Solutions:

For Question 7(c): the basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of part (a) is an orthogonal basis so

$$\begin{aligned}
\vec{u} = \text{proj}_U(\vec{x}) &= \frac{\vec{x}\cdot\vec{v}_1}{\|\vec{v}_1\|^2}\vec{v}_1 + \frac{\vec{x}\cdot\vec{v}_2}{\|\vec{v}_2\|^2}\vec{v}_2 + \frac{\vec{x}\cdot\vec{v}_3}{\|\vec{v}_3\|^2}\vec{v}_3 \\
&= \frac{12}{12}\begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \frac{0}{6}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + \frac{18}{18}\begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 3 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 1 \\ 2 \\ -3 \end{bmatrix}, \text{ as before.}
\end{aligned}$$

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

The end.