University of Toronto
Faculty of Applied Science and Engineering
Final Examination, December 2018
Duration: 2 and 1/2 hrs
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS
Solutions to **MAT188H1F - Linear Algebra**
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Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Observations:

- The range on every question was 0 to 10. Only Question 8 had a failing average, but the averages on Questions 2, 3 and 5 were all (surprisingly) less than 60%. "Surprisingly" because even though part 3(c) *was* challenging, all the other parts of Questions 2, 3 and 5 were completely routine.
- The number of perfect solutions for each question was 190, 73, 13, 322, 8, 323, 299 and 12, respectively; whereas the number of zeros for each question was 3, 114, 46, 78, 27, 33, 20 and 357, respectively.
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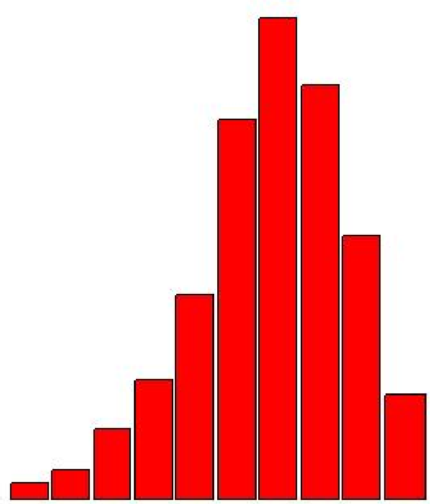
Breakdown of Results: 809 registered students wrote this test. The marks ranged from 1% to 100%, and the average was 62.45%. There was 1 perfect paper. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.6%
A	18.2%	80-89%	12.6%
B	19.8%	70-79%	19.8%
C	23.0%	60-69%	23.0%
D	18.2%	50-59%	18.2%
F	20.8%	40-49%	9.8%
		30-39%	5.7%
		20-29%	3.3%
		10-19%	1.3%
		0-9%	0.7%



1. [avg: 8.09/10] Given that the reduced row echelon form of

$$A = \begin{bmatrix} 1 & 3 & -1 & 7 & -6 \\ 5 & 15 & 4 & 5 & 27 \\ 3 & 9 & 6 & -9 & 39 \\ -6 & -18 & 2 & 5 & -23 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 3 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

find the rank of A and a basis for each of $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$.

Solution:

- by definition, the rank of A is 3, the number of leading 1's in R.
- a basis for $\text{row}(A)$ consists of the three non-zero rows of R,

$$\left\{ \begin{bmatrix} 1 & 3 & 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & -1 \end{bmatrix} \right\},$$

  OR any three *independent* rows of A, which you must demonstrate *are* independent.
- a basis for $\text{col}(A)$ consists of the columns of A that correspond to the columns of R with leading 1's,

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 3 \\ -6 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 5 \\ -9 \\ 5 \end{bmatrix} \right\},$$

  OR any three *independent* columns of A, which you must demonstrate *are* independent.
- a basis for $\text{null}(A)$ consists of the basic solutions to the homogeneous system of equations $A\vec{x} = \vec{0}$, which can be read off R:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

  Aside: if not by inspection then you can get the basic solutions to $A\vec{x} = \vec{0}$ by finding the general solution and writing it as a linear combination:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3s - 4t \\ s \\ -3t \\ t \\ t \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ -3 \\ 1 \\ 1 \end{bmatrix}.$$

  where s and t are parameters.

2. [avg: 5.97/10] For this question, let A be a 5×7 matrix.

(a) [6 marks] Find all possible values of $\dim(\text{col}(A))$ and the corresponding values of $\dim(\text{null}(A))$.

Solution: for this whole question let r be the rank of A. We have

$$\dim(\text{null}(A)) = 7 - r \text{ and } \dim(\text{col}(A)) = \dim(\text{row}(A)) = r.$$

Since A can have at most 5 independent rows, we know $r \leq 5$. Thus the six possibilities are

$$(\dim(\text{col}(A)), \dim(\text{null}(A))) = (0,7), (1,6), (2,5), (3,4), (4,3) \text{ or } (5,2).$$

(b) [4 marks; 1 mark for each part] Now suppose $\dim(\text{null}(A)) = 3$. Find the value of

(i) $\dim(\text{im}(A))$

Solution: we have $r = 7 - \dim(\text{null}(A)) = 4$. Then

$$\dim(\text{im}(A)) = \dim(\text{col}(A)) = r = 4.$$

(ii) $\dim(\text{row}(A))$

Solution: $\dim(\text{row}(A)) = r = 4$.

(iii) $\dim\left((\text{null}(A))^\perp\right)$

Solution: $\dim\left((\text{null}(A))^\perp\right) = 7 - \dim(\text{null}(A)) = 7 - 3 = 4$.

(iv) $\dim\left(\text{null}(A^T)\right)$

Solution: A^T is 7×5 and the rank of A^T is also $r = 4$, so

$$\dim\left(\text{null}(A^T)\right) = 5 - 4 = 1.$$

3. [avg: 5.85/10] Show that the matrices

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -4 & -7 & 2 \\ -12 & -24 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -3 & 1 \\ -4 & -7 & 2 \\ -16 & -30 & 9 \end{bmatrix}$$

(a) [4 marks] have the same characteristic polynomial

Solution: $C_A(x) = (x-1)\det\begin{bmatrix} x+7 & -2 \\ 24 & x-7 \end{bmatrix} = (x-1)(x^2-49+48) = (x-1)^2(x+1)$; and $C_B(x)$

$= \det\begin{bmatrix} x+1 & 3 & -1 \\ 4 & x+7 & -2 \\ 16 & 30 & x-9 \end{bmatrix} = \det\begin{bmatrix} x-1 & 3 & -1 \\ 0 & x+7 & -2 \\ 2x-2 & 30 & x-9 \end{bmatrix} = \det\begin{bmatrix} x-1 & 3 & -1 \\ 0 & x+7 & -2 \\ 0 & 24 & x-7 \end{bmatrix}$

$= (x-1)(x^2-49+48) = (x-1)^2(x+1)$. So indeed it is true that $C_A(x) = C_B(x)$.

(b) [4 marks] and the same rank,

Short solution: since all the eigenvalues of both A and B are non-zero, A and B are both invertible. So each of them has rank 3.

Long solution: you can find that for each matrix, it's reduced row echelon form is I, the 3×3 identity matrix. So the rank of both A and B is 3.

(c) [2 marks] but are *not* similar.

Solution: A is diagonalizable, since

$$\dim(E_1(A)) = \dim\left(\text{null}\begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & -2 \\ 12 & 24 & -6 \end{bmatrix}\right) = \dim\left(\text{null}\begin{bmatrix} 0 & 0 & 0 \\ 2 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right) = 2.$$

But B isn't diagonalizable, since

$$\dim(E_1(B)) = \dim\left(\text{null}\begin{bmatrix} 2 & 3 & -1 \\ 4 & 8 & -2 \\ 16 & 30 & -8 \end{bmatrix}\right) = \dim\left(\text{null}\begin{bmatrix} 2 & 3 & -1 \\ 0 & 2 & 0 \\ 0 & 6 & 0 \end{bmatrix}\right) = 1 < 2.$$

Since A is diagonalizable there is an invertible matrix P such that $D = P^{-1}AP$. If A and B are similar then there is an invertible matrix Q such that $A = Q^{-1}BQ$. Therefore

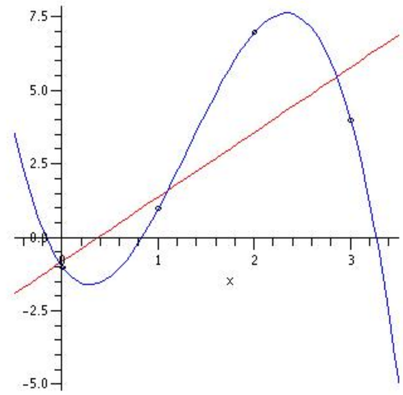
$$D = P^{-1}Q^{-1}BQP = (QP)^{-1}B(QP),$$

which would mean B is diagonalizable as well. Contradiction. So A and B are not similar.

4. [avg: 6.6/10] Consider the four data points $(x, y) = (0, -1), (1, 1), (2, 7), (3, 4)$.

(a) [6 marks] Find the least squares approximating line $y = z_0 + z_1 x$ for the given data points.

Solution: use the normal equations. Let



$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad Z = \begin{bmatrix} z_0 \\ z_1 \end{bmatrix}, \quad Y = \begin{bmatrix} -1 \\ 1 \\ 7 \\ 4 \end{bmatrix}.$$

Solve the normal equations for Z :

$$M^T M Z = M^T Y \Leftrightarrow \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 11 \\ 27 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} z_0 \\ z_1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 11 \\ 27 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 27 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} -8 \\ 42 \end{bmatrix} = \begin{bmatrix} -0.4 \\ 2.1 \end{bmatrix}$$

So the least squares approximating line to the data has equation $y = -0.4 + 2.1\,x$.

(b) [4 marks] Find the cubic polynomial $f(x) = a + bx + cx^2 + dx^3$ such that each of the given data points satisfies the equation $y = f(x)$. (This is *not* a least squares approximating problem.)

Solution: solve the following linear system for a, b, c and d :

$$\left.\begin{array}{r} f(0) = -1 \\ f(1) = 1 \\ f(2) = 7 \\ f(3) = 4 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} a = -1 \\ a + b + c + d = 1 \\ a + 2b + 4c + 8d = 7 \\ a + 3b + 9c + 27d = 4 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{r} b + c + d = 2 \\ b + 2c + 4d = 4 \\ 3b + 9c + 27d = 5 \end{array}\right.$$

You can solve this last system by row reduction on the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 4 \\ 3 & 9 & 27 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 6 & 24 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 6 & -13 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -13/3 \\ 0 & 1 & 0 & 17/2 \\ 0 & 0 & 1 & -13/6 \end{array}\right]$$

So the cubic polynomial that fits the given data points is

$$f(x) = -1 - \frac{13}{3}x + \frac{17}{2}x^2 - \frac{13}{6}x^3.$$

Aside: for interest, see the figure above, which plots both the interpolating cubic polynomial and the least squares approximating line for the four given data points.

5. [avg: 5.76/10]

5.(a) [4 marks] Is the set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ such that $\det \begin{bmatrix} 2 & -1 & a \\ 3 & 1 & b \\ 1 & -1 & c \end{bmatrix} = 0$ a subspace of $\mathbb{R}^3$?

Solution: Yes. Let U be the given set of vectors. You can show U is a subspace in many different ways. Here are two easy ways:

1. $\det \begin{bmatrix} 2 & -1 & a \\ 3 & 1 & b \\ 1 & -1 & c \end{bmatrix} = 2c - b - 3a - a + 2b + 3c = -4a + b + 5c$. So $U = \text{null}[-4\ 1\ 5]$.
2. Since the first two columns of the given matrix are linearly independent,

$$\det \begin{bmatrix} 2 & -1 & a \\ 3 & 1 & b \\ 1 & -1 & c \end{bmatrix} = 0 \Leftrightarrow \text{the columns are dependent} \Leftrightarrow U = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Using the subspace test would be the longest way.

5.(b) [3 marks] Prove that if $\vec{x}$ and $\vec{y}$ are in $\mathbb{R}^n$, then $\|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 = 4\,\vec{x} \cdot \vec{y}$.

Solution:

$$\begin{aligned} \|\vec{x} + \vec{y}\|^2 - \|\vec{x} - \vec{y}\|^2 &= (\vec{x} + \vec{y}) \cdot (\vec{x} + \vec{y}) - (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) \\ &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} - (\vec{x} \cdot \vec{x} - 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y}) \\ &= \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} + \vec{y} \cdot \vec{y} - \vec{x} \cdot \vec{x} + 2\vec{x} \cdot \vec{y} - \vec{y} \cdot \vec{y} \\ &= 4\vec{x} \cdot \vec{y} \end{aligned}$$

5.(c) [3 marks] Prove: if λ is an eigenvalue of the invertible matrix A, then λ^{-1} is an eigenvalue of A^{-1}.

Solution: recall that A is invertible if and only if $\lambda = 0$ is *not* an eigenvalue of A. Let λ be an eigenvalue of A and let $\vec{v} \neq \vec{0}$ be a corresponding eigenvector of A. Then

$$\begin{aligned} A\vec{v} = \lambda\vec{v} &\Rightarrow \vec{v} = A^{-1}(\lambda\vec{v}) \\ &\Rightarrow \vec{v} = \lambda\, A^{-1}\vec{v} \\ &\Rightarrow \frac{1}{\lambda}\vec{v} = A^{-1}\,\vec{v}, \text{ since } \lambda \neq 0 \\ &\Rightarrow \lambda^{-1}\,\vec{v} = A^{-1}\,\vec{v} \end{aligned}$$

Thus λ^{-1} is an eigenvalue of A^{-1} (with the same corresponding eigenvector.)

6. [avg: 7.09/10] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 5 & -2 & -4 \\ -2 & 8 & -2 \\ -4 & -2 & 5 \end{bmatrix}.$$

Step 1: find the eigenvalues of A.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 5 & 2 & 4 \\ 2 & \lambda - 8 & 2 \\ 4 & 2 & \lambda - 5 \end{bmatrix} = \det \begin{bmatrix} \lambda - 9 & 2 & 4 \\ 0 & \lambda - 8 & 2 \\ 9 - \lambda & 2 & \lambda - 5 \end{bmatrix}$$

$$= \det \begin{bmatrix} \lambda - 9 & 2 & 4 \\ 0 & \lambda - 8 & 2 \\ 0 & 4 & \lambda - 1 \end{bmatrix} = (\lambda - 9)(\lambda^2 - 9\lambda + 8 - 8) = \lambda(\lambda - 9)^2$$

Thus the eigenvalues of A are $\lambda_1 = 9$, repeated, and $\lambda_2 = 0$.

Step 2: find an *orthogonal* basis of eigenvectors for each eigenspace.

$$E_9(A) = \text{null} \begin{bmatrix} 9-5 & 2 & 4 \\ 2 & 9-8 & 2 \\ 4 & 2 & 9-5 \end{bmatrix} = \text{null} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ 1 \end{bmatrix} \right\};$$

$$E_0(A) = \text{null} \begin{bmatrix} 0-5 & 2 & 4 \\ 2 & 0-8 & 2 \\ 4 & 2 & 0-5 \end{bmatrix} = \text{null} \begin{bmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & -18 & 9 \\ 1 & -4 & 1 \\ 0 & 18 & -9 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

Step 3: for the columns of P, take the unit, orthogonal eigenvectors and for the diagonal entries of D take the corresponding eigenvalues:

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \\ 0 & -4/\sqrt{18} & 1/3 \\ -1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix} \text{ and } D = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Note: $E_0(A) = (E_9(A))^\perp$, which provides an alternate approach. For instance, with $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ as the basis for $E_0(A)$, and with *any* eigenvector $\vec{v} \in E_9(A)$ you can get an orthogonal basis $\{\vec{v}, \vec{w}\}$ for $E_9(A)$ by taking $\vec{w} = \vec{v} \times \vec{u}$. Try it.

7. [avg: 8.05/10] Let $U = \text{span}\left\{ \begin{bmatrix} 1 & -1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T \right\}$.

(a) [5 marks] Find an orthogonal basis of U.

Solution: use the Gram-Schmidt algorithm. Call the three given vectors $\vec{x}_1, \vec{x}_2, \vec{x}_3$, respectively. Take $\vec{f}_1 = \vec{x}_1$. Then

$$\vec{f}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix},$$

$$\vec{f}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 - \frac{\vec{x}_3 \cdot \vec{f}_2}{\|\vec{f}_2\|^2}\vec{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{4}{15}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 6 \\ -3 \\ -9 \\ 3 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix}.$$

(b) [5 marks] Let $\vec{x} = \begin{bmatrix} 2 & 0 & -1 & 3 \end{bmatrix}^T$. Find $\text{proj}_U(\vec{x})$.

Solution: use the projection formula, for which you must use an orthogonal basis of U :

$$\begin{aligned} \text{proj}_U(\vec{x}) &= \frac{\vec{x} \cdot \vec{f}_1}{\|\vec{f}_1\|^2}\vec{f}_1 + \frac{\vec{x} \cdot \vec{f}_2}{\|\vec{f}_2\|^2}\vec{f}_2 + \frac{\vec{x} \cdot \vec{f}_3}{\|\vec{f}_3\|^2}\vec{f}_3 \\ &= \frac{1}{3}\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \frac{10}{15}\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \frac{10}{15}\begin{bmatrix} 2 \\ -1 \\ -3 \\ 1 \end{bmatrix} \\ &= \frac{1}{15}\begin{bmatrix} 35 \\ 5 \\ -15 \\ 40 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ 1 \\ -3 \\ 8 \end{bmatrix} \end{aligned}$$

Alternate solution: $U^\perp = \text{null}\begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}\right\}.$

Then $\text{proj}_U(\vec{x}) = \vec{x} - \text{proj}_{U^T}(\vec{x}) = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 7 \\ 1 \\ -3 \\ 8 \end{bmatrix}$, as before.

8. [avg: 2.55/10] Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation with matrix $A = \begin{bmatrix} \vec{0} & \vec{e}_1 & \vec{e}_2 & \ldots & \vec{e}_{n-1} \end{bmatrix}$, where $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{n-1}, \vec{e}_n$ are the standard basis vectors of $\mathbb{R}^n$.

(a) [2 marks] If $\vec{x}$ is in $\mathbb{R}^n$, what is $T(\vec{x})$?

Solution: $T(\vec{x}) = A\vec{x} = x_1\vec{0} + x_2\vec{e}_1 + x_3\vec{e}_2 + \cdots + x_n\vec{e}_{n-1}$. That is, $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_{n-1} \\ x_n \end{bmatrix}\right) = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \ldots \\ x_n \\ 0 \end{bmatrix}$.

(b) [2 marks] What is A^2?

Solution: $T\left(T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}\right)\right) = T\left(\begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \\ 0 \end{bmatrix}\right) = \begin{bmatrix} x_3 \\ x_4 \\ \vdots \\ x_n \\ 0 \\ 0 \end{bmatrix}$; so $A^2 = \begin{bmatrix} \vec{0} & \vec{0} & \vec{e}_1 & \vec{e}_2 & \ldots & \vec{e}_{n-2} \end{bmatrix}$.

(c) [4 marks] Let $m \geq 1$ be a whole number. Find a basis for each of $\text{col}(A^m)$ and $\text{null}(A^m)$.

Solution: updated December, 2019:

- for $1 \leq m < n$, the pattern is that $A^m = \begin{bmatrix} \underbrace{\vec{0}\ \vec{0}\ \ldots\ \vec{0}}_{m \text{ times}} & \underbrace{\vec{e}_1\ \vec{e}_2\ \vec{e}_3\ \ldots\ \vec{e}_{n-m}}_{n-m \text{ non-zero cols}} \end{bmatrix}$. So a basis for $\text{col}(A^m)$ is $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \ldots, \vec{e}_{n-m}\}$. Since A^m is already in RREF, the free variables in the homogeneous system $A^m\,\vec{x} = \vec{0}$ are $x_1, x_2, \ldots, x_m$; so a basis for $\text{null}(A^m)$ is $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m\}$.
- if $m \geq n$, then A^m is the zero matrix and

$$\text{col}(A^m) = \{\vec{0}\}, \text{null}(A^m) = \mathbb{R}^n.$$

Thus $\text{col}(A^m)$ has no basis, and a basis for $\text{null}(A^m)$ is $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{n-1}, \vec{e}_n\}$, the standard basis for $\mathbb{R}^n$.

(d) [1 mark] What is the least value of m such that A^m is equal to the zero matrix?

Solution: $m = n$, which follows directly from part (c).

(e) [1 mark] Find the characteristic polynomial of A^m, for $m \geq 1$.

Solution: the characteristic polynomial of A^m is $\det(xI - A^m) = x^n$, since $xI - A^m$ is an upper triangular matrix with every diagonal entry equal to x.

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