University of Toronto
Faculty of Applied Science and Engineering
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Duration: 2 and 1/2 hrs
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

Solutions for **MAT188H1F - Linear Algebra**

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Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

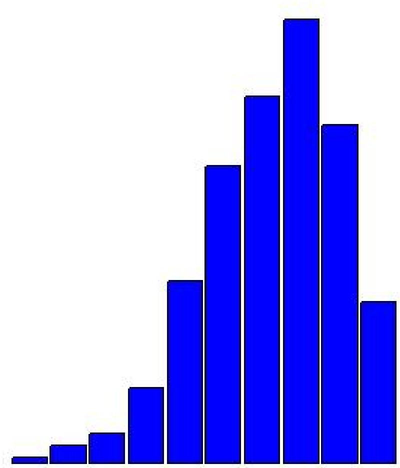
This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. In Question 1, parts (a) and (h) are always true, and part (e) is always false, so none of these three statements can be equivalent to "A is invertible."
2. Questions 3, 5, 6 and 7, similar to questions on previous exams, were well done.
3. Questions 1, 4 and 8, not similar to questions on previous exams, were not so well done.
4. The bonus in Question 8 was only worth one mark.

Breakdown of Results: 948 students wrote this exam. The marks ranged from 0.0% to 101.25%, and the average was 66.9%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	8.4%
A	26.0%	80-89%	17.6%
B	23.1%	70-79%	23.1%
C	19.1%	60-69%	19.1%
D	15.5%	50-59%	15.5%
F	16.2%	40-49%	9.5%
		30-39%	3.9%
		20-29%	1.6%
		10-19%	0.9%
		0-9%	0.3%



PART I : No explanation is necessary.

1. [avg: 3.4/10] Big Theorem, Final Exam Version: Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let

$$A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \ldots & \mathbf{a}_n \end{bmatrix}$$

be the matrix with the vectors in $\mathcal{A}$ as its columns, and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation defined by $T(\mathbf{x}) = A\,\mathbf{x}$. Decide if the following statements are equivalent to the statement, "A is invertible." Circle Yes if the statement is equivalent to "A is invertible," and No if it isn't.

Note: +1 for each correct choice; −1 for each incorrect choice; and 0 for each part left blank.

(a) $\mathcal{A}$ spans col(A). Yes **[No]**

(b) A^T is invertible. **[Yes]** No

(c) The reduced echelon form of A is I, the $n \times n$ identity matrix. **[Yes]** No

(d) T is onto. **[Yes]** No

(e) $\mathbf{0}$ is not in col(A). Yes **[No]**

(f) $\ker(T) = \{\mathbf{0}\}$. **[Yes]** No

(g) $\mathcal{A}$ is a basis for row(A). **[Yes]** No

(h) $\dim(\text{col}(A)) = \dim(\text{row}(A))$. Yes **[No]**

(i) $\text{null}(\text{adj}(A)) = \{\mathbf{0}\}$. **[Yes]** No

(j) $\lambda = 0$ is an eigenvalue of A. Yes **[No]**

PART II : Present **COMPLETE** solutions to the following questions in the space provided.

2. [avg: 7.65/10] Find the following:

(a) [2 marks] $\dim(S^\perp)$, if S is a subspace of $\mathbb{R}^6$ and $\dim(S) = 2$.

Solution:

$$\dim(S^\perp) = 6 - \dim(S) = 6 - 2 = 4.$$

(b) [2 marks] $\det(-2A^2B^T)$, if A and B are 3×3 matrices with $\det(A) = 1$ and $\det(B) = 3$.

Solution:

$$\det(-2A^2B^T) = (-2)^3\,(\det(A))^2\,\det(B) = -8\,(1)^2(3) = -24.$$

(c) [2 marks] all values of a such that $\mathbf{u} = \begin{bmatrix} 1 \\ a \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3a \\ -1 \\ -6 \end{bmatrix}$ are orthogonal.

Solution:

$$\begin{bmatrix} 1 \\ a \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3a \\ -1 \\ -6 \end{bmatrix} = 3a - a - 24 = 2a - 24 = 0 \Rightarrow a = 12.$$

(d) [2 marks] $\det(A)$, if A is an orthogonal matrix.

Solution:

$$A^{-1} = A^T \Rightarrow AA^T = I \Rightarrow \det(AA^T) = \det(I) \Rightarrow (\det(A))^2 = 1 \Rightarrow \det(A) = \pm 1.$$

(e) [2 marks] the dimensions of the square matrix A, if the characteristic polynomial of A is

$$(x-3)^3(x-2)^2(x-1)^4(x+1).$$

Solution: A is $n \times n$, with $n = 3 + 2 + 4 + 1 = 10$.

3. [avg: 7.52/10] Find all values of the parameter a for which the system of equations

$$\begin{array}{rcrcrcl} x_1 & + & ax_2 & + & x_3 & = & 2 \\ x_1 & - & x_2 & + & ax_3 & = & 1 \\ ax_1 & + & x_2 & + & x_3 & = & 1 \end{array}$$

has *(i)* no solution, *(ii)* a unique solution, *(iii)* infinitely many solutions.

Solution: let the coefficient matrix be $A = \begin{bmatrix} 1 & a & 1 \\ 1 & -1 & a \\ a & 1 & 1 \end{bmatrix}$.

$\det(A) = -1+a^3+1+a-a-a = a^3-a = a(a-1)(a+1) = 0 \Leftrightarrow a(a-1)(a+1) = 0 \Leftrightarrow a = 0, -1,$ or 1.

If the coefficient matrix A is invertible, the system of equations will have a unique solution. Therefore, *(ii)*, the system has a unique solution if $a \neq 0, a \neq -1, a \neq 1$.

Now see what happens for $a = 0, -1$ or 1:

(iii): for $a = 0$ the system has infinitely many solutions, $x_1 = 2 - t, x_2 = 1 - t, x_3 = t$:

$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

(i): if $a = \pm 1$ the system has no solution, which can be seen by reducing the augmented matrix:

$$\text{for } a = 1: \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & 1 & 1 & 2 \\ 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right];$$

$$\text{for } a = -1: \left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 1 & 2 \\ 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{array}\right].$$

Alternate Soluiton: find an echelon form of the augmented matrix:

$$\left[\begin{array}{rrr|r} 1 & a & 1 & 2 \\ 1 & -1 & a & 1 \\ a & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & a & 1 & 2 \\ 0 & -1-a & a-1 & -1 \\ 0 & 1-a^2 & 1-a & 1-2a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & a & 1 & 2 \\ 0 & 1+a & 1-a & 1 \\ 0 & 0 & a(1-a) & -a \end{array}\right],$$

and analyze the cases when $a(1-a)(1+a) = 0$ or $a(1-a)(1+a) \neq 0$.

4. [avg: 5.73/10] Let $\mathbf{u} = [\ 1 \quad -2 \quad 3\ ]^T$; let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by $T(\mathbf{x}) = \text{proj}_{\mathbf{u}}\mathbf{x}$.

(a) [5 marks] Find the matrix of T.

Solution: let $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$; let the matrix of T be A. Then

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix} = \frac{1}{\|\mathbf{u}\|^2} \begin{bmatrix} (\mathbf{e}_1 \cdot \mathbf{u})\,\mathbf{u} & (\mathbf{e}_2 \cdot \mathbf{u})\,\mathbf{u} & (\mathbf{e}_3 \cdot \mathbf{u})\,\mathbf{u} \end{bmatrix} = \frac{1}{14} \begin{bmatrix} \mathbf{u} & -2\mathbf{u} & 3\mathbf{u} \end{bmatrix};$$

that is,

$$A = \frac{1}{14} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix}.$$

Alternate Solution: use properties of dot product and matrix multiplication.

$$\text{proj}_{\mathbf{u}}\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{u}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{x}}{\|\mathbf{u}\|^2}\mathbf{u} = \frac{\mathbf{u}\,(\mathbf{u}^T\mathbf{x})}{\|\mathbf{u}\|^2} = \frac{(\mathbf{u}\,\mathbf{u}^T)\,\mathbf{x}}{\|\mathbf{u}\|^2} = \frac{1}{\|\mathbf{u}\|^2}(\mathbf{u}\,\mathbf{u}^T)\,\mathbf{x} \Rightarrow A = \frac{1}{\|\mathbf{u}\|^2}(\mathbf{u}\,\mathbf{u}^T).$$

(b) [5 marks] Find a basis for each of $\ker(T)$ and $\text{range}(T)$.

Solution: $\text{range}(T) = \text{col}(A) = \text{span}\{\mathbf{u}, -2\mathbf{u}, 3\mathbf{u}\} = \text{span}\{\mathbf{u}\}$; so

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$$

is a basis for $\text{range}(T)$.

$$\ker(T) = \text{null}(A) = \text{null} \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -6 \\ 3 & -6 & 9 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\};$$

so

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\ker(T)$, since the two spanning vectors are independent.

Aside: $\ker(T) = (\text{range}(T))^{\perp}$.

5. [avg: 7.86/10] Find the solution to the system of linear differential equations

$$\begin{array}{rcrcr} y_1' &=& 2y_1 &-& y_2 \\ y_2' &=& 6y_1 &-& 5y_2 \end{array}$$

where y_1, y_2 are functions of t, and $y_1(0) = 2,\ y_2(0) = 3$.

Solution: let the coefficient matrix be $A = \begin{bmatrix} 2 & -1 \\ 6 & -5 \end{bmatrix}$. We need the eigenvalues and eigenvectors of A.

$$\det(\lambda I - A) = (\lambda - 2)(\lambda + 5) + 6 = \lambda^2 + 3\lambda - 4 = (\lambda + 4)(\lambda - 1) = 0 \Rightarrow \lambda = 1, -4.$$

For the eigenvalue $\lambda_1 = 1$:

$$(\lambda_1 I_2 - A|\mathbf{0}) = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -6 & 6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{so take } \mathbf{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = -4$:

$$(\lambda_2 I_2 - A|\mathbf{0}) = \left[\begin{array}{cc|c} -6 & 1 & 0 \\ -6 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 6 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{so take } \mathbf{u_2} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}.$$

Then the general solution to the system of differential equations is

$$\mathbf{y} = c_1 \mathbf{u}_1 e^t + c_2 \mathbf{u}_2 e^{-4t}.$$

To find c_1, c_2 use the initial conditions, with $t = 0$:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 6 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 6 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 9 \\ 1 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{9}{5} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + \frac{1}{5} \begin{bmatrix} 1 \\ 6 \end{bmatrix} e^{-4t}$$

and

$$y_1 = \frac{9}{5} e^t + \frac{1}{5} e^{-4t};\ y_2 = \frac{9}{5} e^t + \frac{6}{5} e^{-4t}.$$

6. [avg: 6.71/10] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda-5 & 2 & -4 \\ 2 & \lambda-8 & -2 \\ -4 & -2 & \lambda-5 \end{bmatrix} = \det \begin{bmatrix} \lambda-5 & 2 & \lambda-9 \\ 2 & \lambda-8 & 0 \\ -4 & -2 & \lambda-9 \end{bmatrix} \\ &= \det \begin{bmatrix} \lambda-1 & 4 & 0 \\ 2 & \lambda-8 & 0 \\ -4 & -2 & \lambda-9 \end{bmatrix} = (\lambda-9)((\lambda-1)(\lambda-8)-8) = \lambda(\lambda-9)^2 \end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 9$, repeated.

Step 2: Find three mutually **orthogonal** eigenvectors of A. Notation: let

$$E_\lambda(A) = \{\mathbf{x} \in \mathbb{R}^3 \mid A\,\mathbf{x} = \lambda\,\mathbf{x}\} = \text{null}(A - \lambda I)$$

be the eigenspace of A corresponding to the eigenvalue λ.

$$E_0(A) = \text{null}(A) = \text{null} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -4 & -1 \\ 0 & 18 & 9 \\ 0 & 18 & 9 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

$$E_9(A) = \text{null} \begin{bmatrix} -4 & -2 & 4 \\ -2 & -1 & 2 \\ 4 & 2 & -4 \end{bmatrix} = \text{null} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -4 \\ -1 \end{bmatrix} \right\}.$$

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \\ -1/3 & 0 & -4/\sqrt{18} \\ 2/3 & -1/\sqrt{2} & -1/\sqrt{18} \end{bmatrix}; \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

7. [avg: 8.54/10] Let $S = \text{span}\left\{[\ 1\ \ 1\ \ 0\ \ 1\ ]^T, [\ 0\ \ 1\ \ -1\ \ 1\ ]^T, [\ 2\ \ 0\ \ 1\ \ 1\ ]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

Solution: call the three given vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and apply the Gram-Schmidt algorithm to find an orthogonal basis $\mathbf{f}_2, \mathbf{f}_2, \mathbf{f}_3$. Take $\mathbf{f}_1 = \mathbf{x}_1$,

$$\mathbf{f}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix},$$

$$\mathbf{f}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{6}{15}\begin{bmatrix} -2 \\ 1 \\ -3 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix}.$$

Alternate Solution: observe that $\mathbf{x}_2 \cdot \mathbf{x}_3 = 0$, so take $\mathbf{f}_1 = \mathbf{x}_2,\ \mathbf{f}_2 = \mathbf{x}_3$, and

$$\mathbf{f}_3 = \mathbf{x}_1 - \frac{\mathbf{x}_1 \cdot \mathbf{x}_2}{\|\mathbf{x}_2\|^2}\mathbf{x}_2 - \frac{\mathbf{x}_1 \cdot \mathbf{x}_3}{\|\mathbf{x}_3\|^2}\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{3}{6}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

Either way $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}^T$. Find $\text{proj}_S(\mathbf{x})$.

Solution: use the projection formula.

$$\text{proj}_S\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 = \frac{7}{3}\mathbf{f}_1 - \frac{5}{15}\mathbf{f}_2 + \frac{0}{15}\mathbf{f}_3 = [\ 3\ \ 2\ \ 1\ \ 2\ ]^T.$$

Alternate Solution:

$$\text{proj}_S\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 = \frac{3}{3}\mathbf{f}_1 + \frac{9}{6}\mathbf{f}_2 + 3\mathbf{f}_3 = [\ 3\ \ 2\ \ 1\ \ 2\ ]^T.$$

Cross-check/Other Solution: $S^\perp = \text{span}\{\mathbf{y}\}$ with $\mathbf{y} = [\ -1\ \ 0\ \ 1\ \ 1\ ]^T$. Then

$$\text{proj}_S\mathbf{x} = \mathbf{x} - \text{proj}_{S^\perp}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} = \mathbf{x} - \frac{6}{3}\mathbf{y} = [\ 3\ \ 2\ \ 1\ \ 2\ ]^T.$$

8. [avg: 6.05/11] Let $A = \begin{bmatrix} 1/3 & 3/4 \\ 2/3 & 1/4 \end{bmatrix}$.

(a) [6 marks] Find an invertible matrix P and a diagonal matrix D such that $D = P^{-1}AP$.

Solution:

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 1/3 & -3/4 \\ -2/3 & \lambda - 1/4 \end{bmatrix} = \lambda^2 - \frac{7}{12}\lambda - \frac{5}{12} = (\lambda - 1)(\lambda + 5/12).$$

The eigenvalues of A are

$$\lambda_1 = 1 \text{ and } \lambda_2 = -\frac{5}{12}.$$

Find the eigenvectors:

$$\text{null}(\lambda_1 I - A) = \text{null} \begin{bmatrix} 2/3 & -3/4 \\ -2/3 & 3/4 \end{bmatrix} = \text{null} \begin{bmatrix} 8 & -9 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 9 \\ 8 \end{bmatrix} \right\};$$

$$\text{null}(\lambda_2 I - A) = \text{null} \begin{bmatrix} -3/4 & -3/4 \\ -2/3 & -2/3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}.$$

Take

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -5/12 \end{bmatrix} \text{ and } P = \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix}.$$

(b) [4 marks] Find and simplify A^n. (Bonus: what can you say about the entries of A^n as $n \to \infty$?)

Solution:

$$\begin{aligned} A^n = PD^nP^{-1} &= \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -5/12 \end{bmatrix}^n \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-5/12)^n \end{bmatrix} \frac{1}{17} \begin{bmatrix} 1 & 1 \\ -8 & 9 \end{bmatrix} \\ &= \frac{1}{17} \begin{bmatrix} 9 & -1 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -8(-5/12)^n & 9(-5/12)^n \end{bmatrix} \\ &= \frac{1}{17} \begin{bmatrix} 9 + 8(-5/12)^n & 9 - 9(-5/12)^n \\ 8 - 8(-5/12)^n & 8 + 9(-5/12)^n \end{bmatrix} \end{aligned}$$

Bonus: as $n \to \infty$,

$$A^n \to \frac{1}{17} \begin{bmatrix} 9 & 9 \\ 8 & 8 \end{bmatrix}.$$

This page is for rough work; it will not be marked.

The end.