University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2010**
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

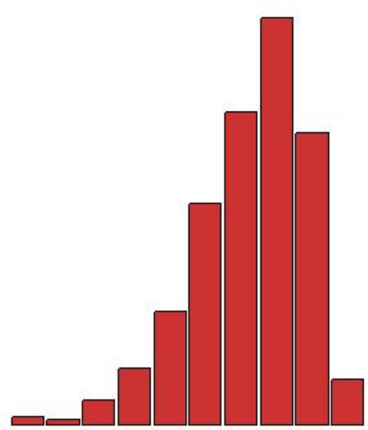
MAT188H1F - LINEAR ALGEBRA
Exam Type: A

General Comments:

1. Questions 1(a), 1(c), 1(d) and 5 are all completely routine, and should have been aced. Questions 3 4, 6 and 7 were the same types of questions as found on last year's exam. These questions should also have been aced!
2. In Question 1(b) you have to remember that there are *three* possible types of solution to a linear system of equations: a unique solution, no solution, or infinitely many solutions. So only finding the values of a for which the coefficient matrix is not invertible, is *not* good enough.
3. For Question 4 some students misread the question and tried to find the least squares approximating *quadratic* which is a much messier problem.
4. The matrix in Question 7 was the same as on the term test.
5. The first two parts of Question 8 are computations with matrices that could have been done in Chapter 1. But many students made incorrect simplifications. A is *not* square! P_A is *not* I. $(AB)^T \neq A^T B^T$; $(AB)^{-1} \neq A^{-1}B^{-1}$. Consequently, very few students got the *easy* 6 marks in this question .... Only one student solved Question 8 completely.

Breakdown of Results: 946 students wrote this exam. The marks ranged from 2% to 99%, and the average was 66.8%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.1%
A	22.8%	80-89%	19.7%
B	27.4%	70-79%	27.4%
C	21.0%	60-69%	21.0%
D	14.9%	50-59%	14.9%
F	13.9%	40-49%	7.6%
		30-39%	3.8%
		20-29%	1.7%
		10-19%	0.3%
		0-9%	0.5%



1. Find the following:

   (a) [6 marks] the characteristic polynomial of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   Solution:

$$C_A(x) = \det \begin{bmatrix} x-a & -b \\ -c & x-d \end{bmatrix} = x^2 - (a+d)\,x + ad - bc.$$

   (b) [6 marks] all values of a for which the following system of equations has

$$\begin{cases} x + y + z = 1 \\ -x + ay = 3 \\ 6y + az = 8 \end{cases}$$

   infinitely many solutions.

   Solution:

$$\det \begin{bmatrix} 1 & 1 & 1 \\ -1 & a & 0 \\ 0 & 6 & a \end{bmatrix} = a^2 - 6 + a = (a+3)(a-2) = 0 \Leftrightarrow a = 2 \text{ or } a = -3.$$

   If $a = 2$ the system has infinitely many solutions:

$$\begin{cases} x + y + z = 1 \\ -x + 2y = 3 \\ 6y + 2z = 8 \end{cases} \Rightarrow \begin{cases} x + y + z = 1 \\ 3y + z = 4 \\ 3y + z = 4 \end{cases}$$

   But if $a = -3$ the system is inconsistent:

$$\begin{cases} x + y + z = 1 \\ -x - 3y = 3 \\ 6y - 3z = 8 \end{cases} \Rightarrow \begin{cases} x + y + z = 1 \\ -2y + z = 4 \\ -2y + z = -4 \end{cases}$$

   So the only value of a for which the system has infinitely many solutions is $a = 2$.

(c) [6 marks] a linear combination of the three vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ equal to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution:

$$\left[\begin{array}{rrr|r} 1 & 1 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 1 & -1 & 2 & 3 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 1 & 3 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & 1 & -2 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right];$$

thus

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}.$$

(d) [6 marks] a basis for $U^\perp$ if $U = \text{span}\left\{[\,1\ \ 0\ \ 1\ \ 2\ \ 1\,]^T, [\,1\ \ 0\ \ 2\ \ 1\ \ 1\,]^T, [\,1\ \ 1\ \ 2\ \ 1\ \ 1\,]^T\right\}$.

Solution:

$$\begin{aligned} U^\perp &= \text{null} \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix} \\ &= \text{null} \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{bmatrix} \\ &= \text{null} \begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \\ &= \left\{ \begin{bmatrix} -3s - t \\ 0 \\ s \\ s \\ t \end{bmatrix} \middle|\, s, t \in \mathbb{R} \right\} \end{aligned}$$

So a basis for $U^\perp$ is

$$\left\{ \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2. [2 marks each] Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) $\dim\left(\operatorname{im}\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 3 & 3 & 0 \\ 3 & 1 & 7 & 3 \\ 2 & 1 & 5 & 2 \end{bmatrix}\right) = 2$ **True** or **False**

True:

$$A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 3 & 3 & 0 \\ 3 & 1 & 7 & 3 \\ 2 & 1 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -5 & -5 & 0 \\ 0 & -3 & -3 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}; \operatorname{rank}(A) = 2.$$

(b) If U is a subspace of $\mathbb{R}^5$ and $\dim U = 2$ then $\dim U^\perp = 2$. **True** or **False**

False: $5 = \dim U + \dim U^\perp \Rightarrow \dim U^\perp = 5 - 2 = 3.$

(c) If A is a diagonalizable matrix then so is A^T. **True** or **False**

True:

$$D = P^{-1}AP \Leftrightarrow D^T = \left(P^{-1}AP\right)^T \Leftrightarrow D = P^T A^T (P^{-1})^T \Leftrightarrow D = P^T A^T (P^T)^{-1}.$$

(d) If $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 6x - 5y \\ 3x - 4y \end{bmatrix}$ then the area of the image of the unit square is 9.

True or **False**

True:

$$\left|\det \begin{bmatrix} 6 & -5 \\ 3 & -4 \end{bmatrix}\right| = |-24 + 15| = 9$$

(e) If T_1 is a rotation of $180°$ and T_2 is a reflection in the line $y = 2x$, then $T_1 \circ T_2$ is a projection on the line $y = -2x$. **True** or **False**

False: Rotations and reflections are invertible, but projections aren't.

3. [10 marks] Given that the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & 1 & 3 & 3 & -1 \\ 2 & -1 & 4 & 3 & 0 \\ 4 & 1 & 10 & 9 & -2 \\ 1 & 1 & 1 & 3 & 1 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

Solution: the rank of A is 3, the number of leading 1's in R.

subspace	description of basis	vectors in basis
row$A$	three non-zero rows of R	$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \right\}$
	any three independent rows of A	$\{R_1, R_2, R_4\}$ $\{R_1, R_3, R_4\}$ $\{R_2, R_3, R_4\}$ NB: $R_3 = 2R_1 + R_2$
col A	three independent columns of A	$\{C_1, C_2, C_3\}$ NB: $C_4 = 2C_1 + C_2; C_5 = 2C_1 - C_3$
null A	two basic solutions to $AX = 0$	$\left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

4. [10 marks] Find the least squares approximating line for the data points

$$(-2, 0), (0, 1), (1, 1), (1, 2), (2, 3).$$

Solution: Let the least squares approximating line be $y = a + bx$; let

$$M = \begin{bmatrix} 1 & -2 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}, Z = \begin{bmatrix} a \\ b \end{bmatrix}, Y = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

The normal equations are:

$$M^T M Z = M^T Y \Leftrightarrow \begin{bmatrix} 5 & 2 \\ 2 & 10 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \end{bmatrix}$$

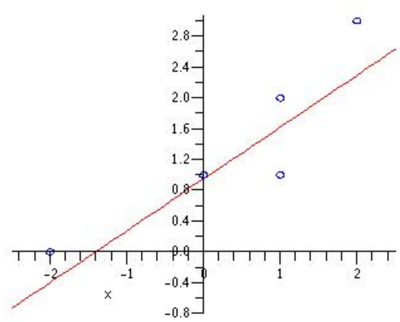
So

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{46} \begin{bmatrix} 10 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 9 \end{bmatrix} = \frac{1}{46} \begin{bmatrix} 52 \\ 31 \end{bmatrix}$$

Answer: the least squares approximating line is

$$y = \frac{26}{23} + \frac{31}{46} x.$$

For interest, the graph is shown to the right.



For Those Who Misread the Question and found the least squares approximating quadratic instead: the normal equations are

$$\begin{bmatrix} 5 & 2 & 10 \\ 2 & 10 & 2 \\ 10 & 2 & 34 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 15 \end{bmatrix}$$

and the the least squares approximating quadratic is

$$y = a + bx + cx^2 = \frac{10}{13} + \frac{37}{52} x + \frac{9}{52} x^2.$$

5. [10 marks] Find the general solution to the following system of differential equations:

$$\begin{aligned} f_1'(x) &= & & & & & f_3(x) \\ f_2'(x) &= & f_1(x) & - & f_2(x) & + & f_3(x) \\ f_3'(x) &= & 4f_1(x) & & & & \end{aligned}$$

Solution: let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & -1 \\ -1 & \lambda+1 & -1 \\ -4 & 0 & \lambda \end{bmatrix} = (\lambda+1)\det \begin{bmatrix} \lambda & -1 \\ -4 & \lambda \end{bmatrix} = (\lambda+1)(\lambda^2-4)$$

So the eigenvalues of A are $\lambda_1 = -1, \lambda_2 = -2$ and $\lambda_3 = 2$.

Eigenvectors:

$$E_{-1}(A) = \text{null} \begin{bmatrix} -1 & 0 & -1 \\ -1 & 0 & -1 \\ -4 & 0 & -1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

$$E_{-2}(A) = \text{null} \begin{bmatrix} -2 & 0 & -1 \\ -1 & -1 & -1 \\ -4 & 0 & -2 \end{bmatrix} = \text{null} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}.$$

$$E_{2}(A) = \text{null} \begin{bmatrix} 2 & 0 & -1 \\ -1 & 3 & -1 \\ -4 & 0 & 2 \end{bmatrix} = \text{null} \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$$

The general solution is

$$\begin{bmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{bmatrix} = c_1 e^{-x} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-2x} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} + c_3 e^{2x} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

6. [12 marks] Let $U = \text{span}\left\{ \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^T, \begin{bmatrix} 2 & 0 & -1 & -1 \end{bmatrix}^T \right\}$; let $X = \begin{bmatrix} 1 & 1 & 2 & 1 \end{bmatrix}^T$. Find:

(a) [6 marks] an orthogonal basis of U.

Solution: Use the Gram-Schmidt algorithm.

$$F_1 = X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}; F_2 = X_2 - \frac{X_2 \cdot F_1}{\|F_1\|^2} F_1 = X_2 + \frac{1}{2} F_1 = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix};$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_3 \cdot F_2}{\|F_2\|^2} F_2 = X_3 + \frac{1}{2} F_1 - \frac{5}{3} F_2 = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix}.$$

Optional: clear fractions and take

$$F_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, F_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \end{bmatrix}, F_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \end{bmatrix}.$$

Either way $\{F_1, F_2, F_3\}$ is an orthogonal basis of U.

(b) [6 marks] $\text{proj}_U(X)$.

Solution: using $\{F_1, F_2, F_3\}$ with fractions cleared.

$$\begin{aligned} \text{proj}_U X &= \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \frac{X \cdot F_2}{\|F_2\|^2} F_2 + \frac{X \cdot F_3}{\|F_3\|^2} F_3 \\ &= \frac{3}{2} F_1 + \frac{1}{6} F_2 - \frac{1}{12} F_3 \\ &= \frac{1}{4} \begin{bmatrix} 1 \\ 7 \\ 5 \\ 1 \end{bmatrix}. \end{aligned}$$

Cross-check/Alternate Solution: $U^\perp = \text{span}\{Y\}$ with $Y = \begin{bmatrix} 1 & -1 & 1 & 1 \end{bmatrix}^T$. Then

$$\text{proj}_U X = X - \text{proj}_{U^\perp}(X) = X - \frac{X \cdot Y}{\|Y\|^2} Y = X - \frac{3}{4} Y = \frac{1}{4} \begin{bmatrix} 1 \\ 7 \\ 5 \\ 1 \end{bmatrix}.$$

7. [12 marks] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\begin{aligned}
\det(\lambda I - A) &= \det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} = \det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ 0 & -1-\lambda & \lambda+1 \end{bmatrix} \\
&= (\lambda+1)\det \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ 0 & -1 & 1 \end{bmatrix} = (\lambda+1)\det \begin{bmatrix} \lambda & -2 & -1 \\ -1 & \lambda-1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= (\lambda+1)\left(\lambda(\lambda-1)-2\right) = (\lambda+1)(\lambda^2-\lambda-2) \\
&= (\lambda+1)^2(\lambda-2)
\end{aligned}$$

So the eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 2$.

Step 2: Find mutually orthogonal eigenvectors of A.

$$E_2(A) = \text{null} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$E_{-1}(A) = \text{null} \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

OR: use either $E_{-1}(A) = (E_2(A))^\perp$ or $E_2(A) = (E_{-1}(A))^\perp$ to simplify calculations.

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

8.(a) [3 marks] Let $A = \begin{bmatrix} 1 \\ m \end{bmatrix}$. Show that $P = A\,(A^T A)^{-1} A^T$ is the matrix of a projection.

Solution:

$$\begin{bmatrix} 1 \\ m \end{bmatrix} \left(\begin{bmatrix} 1 \\ m \end{bmatrix}^T \begin{bmatrix} 1 \\ m \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ m \end{bmatrix}^T = \frac{1}{1+m^2} \begin{bmatrix} 1 \\ m \end{bmatrix} \begin{bmatrix} 1 & m \end{bmatrix} = \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}.$$

8.(b) [9 marks] Let U be a subspace of $\mathbb{R}^n$ with basis $X_1, X_2, \ldots, X_k$; let A be the $n \times k$ matrix with columns $X_1, X_2, \ldots, X_k$. Let $P_A = A\,(A^T A)^{-1} A^T$.

(*i*) [3 marks] Show each of the following:

$P_A^2 = P_A$:

Solution: $P_A^2 = A\,(A^T A)^{-1} (A^T A)\,(A^T A)^{-1} A^T = A\,(A^T A)^{-1} A^T = P_A$

$P_A^T = P_A$:

Solution:

$$P_A^T = \left(A\,(A^T A)^{-1} A^T\right)^T = (A^T)^T \left((A^T A)^{-1}\right)^T A^T = A\,(A^T A)^{-1} A^T = P_A$$

$P_A\,A = A$:

Solution: $P_A\,A = A\,(A^T A)^{-1} A^T\,A = A\,(A^T A)^{-1} (A^T\,A) = A$

(ii) [6 marks] Show that $\operatorname{im}(P_A) = U$ and $\operatorname{null}(P_A) = U^\perp$.

Solution: $P_A A = A \Rightarrow P_A X_i = X_i \Rightarrow U \subset \operatorname{im}(P_A)$. On the other hand,

$$P_A X = \left(A\,(A^T A)^{-1} A^T\right) X = A\left((A^T A)^{-1} A^T X\right) \Rightarrow \operatorname{im}(P_A) \subset \operatorname{im}(A) = U.$$

Thus $\operatorname{im}(P_A) = U$.

$X \in \operatorname{null}(A^T) \Rightarrow A^T(X) = O \Rightarrow P_A(X) = A\,(A^T A)^{-1} (A^T X) = O$, so we must have

$$U^\perp = \operatorname{null}(A^T) \subset \operatorname{null}(P_A).$$

But

$$\dim \operatorname{null}(P_A) = n - \dim \operatorname{im}(P_A) = n - \dim U = \dim U^\perp,$$

so $\operatorname{null}(P_A) = U^\perp$.

Alternately: Since $V = \operatorname{col}(B) \Leftrightarrow V^\perp = \operatorname{null}(B^T)$, and P_A is symmetric, $U = \operatorname{im}(P_A) \Leftrightarrow U^\perp = \operatorname{null}(P_A)$; so you only have to prove one of the two directly.