University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, DECEMBER, 2009**
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

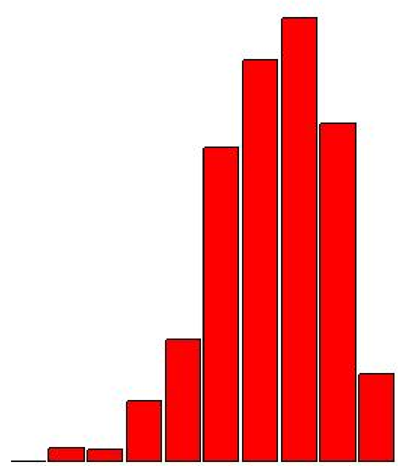
MAT188H1F - LINEAR ALGEBRA
Exam Type: A

General Comments:

1. Most of this exam was very similar to last year's exam. To be specific, in Questions 8, 11, 12 and 13 only the numbers were changed. Anybody who studied last year's exam should have aced these four questions.
2. The only question on this exam which is not completely routine is Question 9(b), for which two solutions are included. Question 9(b) is the hard 5% of the exam.
3. For some reason many students completely misread Question 10: instead of finding a basis of $U^\perp$, many students found an orthogonal basis of U.
4. In Question 6 many students circled a negative answer for area, which cannot be right since area is positive.

Breakdown of Results: 897 students wrote this exam. The marks ranged from 11% to 97%, and the average was 67.3 %. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.9%
A	23.7%	80-89%	18.8%
B	24.8%	70-79%	24.8%
C	22.4%	60-69%	22.4%
D	17.5%	50-59%	17.5%
F	11.6%	40-49%	6.8%
		30-39%	3.3%
		20-29%	0.7%
		10-19%	0.8%
		0-9%	0.0%



1. The values of a for which the matrix $\begin{bmatrix} 6 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & 6 \end{bmatrix}$ is not invertible, are

   (a) $a = \pm 3$

   (b) $a = \pm 2$

   (c) $a = \pm 6$

   (d) $a = \pm 9$

Solution:

$$\det \begin{bmatrix} 6 & 0 & a \\ 0 & 2 & 0 \\ a & 0 & 6 \end{bmatrix} = 2(36 - a^2) = 0 \Leftrightarrow a^2 = 36$$

The answer is (c).

2. If A and B are 3×3 matrices such that $\det(A) = 2$ and $\det(B) = -3$, then the value of $\det(A^2 B^{-1} A^T B^3)$ is

   (a) -72

   (b) 72

   (c) -108

   (d) 108

Solution:

$$\begin{aligned} \det(A^2 B^{-1} A^T B^3) &= (\det A)^2 (\det B)^{-1} (\det A)(\det B)^3 \\ &= (\det A)^3 (\det B)^2 \\ &= 8 \times 9 = 72 \end{aligned}$$

The answer is (b).

3. The characteristic polynomial of the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

   (a) $C_A(x) = x^2 - (a+b)x + c + d$

   (b) $C_A(x) = x^2 - (a+d)x + cb - ad$

   (c) $C_A(x) = x^2 + (a+b)x + ad - bc$

   (d) $C_A(x) = x^2 - (a+d)x + ad - bc$

Solution:

$$\begin{aligned} C_A(x) &= \det \begin{bmatrix} x-a & -b \\ -c & x-d \end{bmatrix} \\ &= x^2 - (a+d)x + ad - bc \end{aligned}$$

The answer is (d).

4. If $\vec{u} \cdot \vec{v} \times \vec{w} = 10$, then the value of $3\vec{v} \cdot \vec{u} \times \vec{w}$ is

(a) 10

(b) 30

(c) -10.

(d) -30

Solution:

$$\begin{aligned} 3\vec{v} \cdot \vec{u} \times \vec{w} &= \det[\ 3\vec{v} \mid \vec{u} \mid \vec{w}\ ] \\ &= 3\det[\ \vec{v} \mid \vec{u} \mid \vec{w}\ ] \\ &= -3\det[\ \vec{u} \mid \vec{v} \mid \vec{w}\ ] \\ &= (-3) \times (10) \end{aligned}$$

The answer is (d).

5. Suppose two lines in space are both parallel to the vector $\vec{d}$, one line passes through the point X_1, and the other line passes through the point X_2. The minimum distance between these two parallel lines is given by

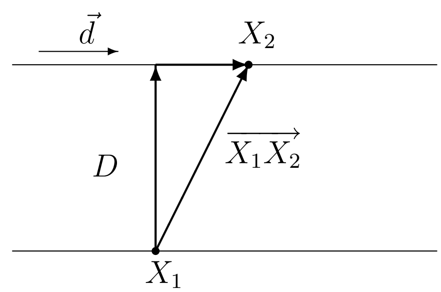
(a) $\left\|\overrightarrow{X_1X_2}\right\|$

(b) $\left\|\text{proj}_{\vec{d}}\left(\overrightarrow{X_1X_2}\right)\right\|$

(c) $\sqrt{\left\|\overrightarrow{X_1X_2}\right\|^2 - \left\|\text{proj}_{\vec{d}}\left(\overrightarrow{X_1X_2}\right)\right\|^2}$

(d) $\sqrt{\left\|\text{proj}_{\vec{d}}\left(\overrightarrow{X_1X_2}\right)\right\|^2 - \left\|\overrightarrow{X_1X_2}\right\|^2}$

Solution:



$$D^2 + \left\|\text{proj}_{\vec{d}}\left(\overrightarrow{X_1X_2}\right)\right\|^2 = \left\|\overrightarrow{X_1X_2}\right\|^2$$

The answer is (c).

6. Suppose $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 7y \\ 8x + 6y \end{bmatrix}$. Then the area of the image of the unit square is

(a) -68

(b) 68

(c) -44

(d) 44

Solution:

$$\left|\det \begin{bmatrix} 2 & 7 \\ 8 & 6 \end{bmatrix}\right| = |12 - 56| = 44.$$

The answer is (d).

7. Decide if the following statements are True or False, and give a brief, concise justification for your choice. Circle your choice.

(a) $\operatorname{span}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix},\begin{bmatrix}1\\4\\3\end{bmatrix},\begin{bmatrix}3\\1\\1\end{bmatrix},\begin{bmatrix}5\\1\\6\end{bmatrix}\right\}=\mathbb{R}^3.$ **True** or **False**

> **True:** $\det\begin{bmatrix}1&1&3\\0&4&1\\1&3&1\end{bmatrix}=-10\neq 0$, so the first three given vectors span $\mathbb{R}^3$.

(b) If U is a subspace of $\mathbb{R}^6$ and $\dim U=4$ then $\dim U^\perp=2$. **True** or **False**

> **True:** $6=\dim U+\dim U^\perp \Rightarrow \dim U^\perp = 6-4=2.$

(c) If A is an invertible matrix then $\lambda=0$ is an eigenvalue of A. **True** or **False**

> **False:** if A is invertible, then $\text{null}(A)=\{0\}$, so there is no non-zero vector X such that $AX=0X=0$.

(d) If $\{X,Y,Z\}$ is linearly independent, then $\{X,Y\}$ is also linearly independent. **True** or **False**

> **True:**
>
> $$\begin{aligned} aX+bY=0 &\;\Rightarrow\; aX+bY+0\,Z=0\\ &\;\Rightarrow\; a=b=0\end{aligned}$$

(e) For any $n\times n$ invertible matrix A, $\operatorname{adj}(A^{-1})=(\operatorname{adj}A)^{-1}$. **True** or **False**

> **True:**
>
> $$A\operatorname{adj}A=\det(A)\,I\Rightarrow(\operatorname{adj}A)^{-1}=\frac{1}{\det(A)}A$$
>
> and also
>
> $$A^{-1}\operatorname{adj}(A^{-1})=\det(A^{-1})I\Rightarrow\operatorname{adj}(A^{-1})=\det(A^{-1})\,A=\frac{1}{\det(A)}A.$$

8. Given that the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & -1 & 1 & 3 & 1 \\ 1 & -1 & 2 & 0 & 1 \\ 2 & -2 & 3 & 3 & 2 \\ -1 & 1 & -1 & -1 & 1 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & -1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

Solution: the rank of A is 3, the number of leading 1's in R.

subspace	description of basis	vectors in basis
row$A$	three non-zero rows of R	$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$
	any three independent rows of A	$\{R_1, R_2, R_4\}$ $\{R_1, R_3, R_4\}$ $\{R_2, R_3, R_4\}$ NB: $R_3 = R_1 + R_2$
col A	three independent columns of A	$\{C_1, C_3, C_4\}$ NB: $C_2 = -C_1; C_5 = -5C_1 + 3C_3 + C_4$
null A	two basic solutions to $AX = 0$	$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ -3 \\ -1 \\ 1 \end{bmatrix} \right\}$

9. Let A be an $n \times n$ matrix. Let $U = \{X \in \mathbb{R}^n \mid A^2X = AX\}$.

(a) [5 marks] Show that U is a subspace of $\mathbb{R}^n$.

Solution: $U = \text{null}(A^2 - A)$, so it is a subspace of $\mathbb{R}^n$.

(b) [5 marks] Show that if A is diagonalizable then $\dim U = \dim E_0(A) + \dim E_1(A)$.

Solution: This is quite tricky. Since A is diagonalizable there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$ and $A^2 = PD^2P^{-1}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of A, including repetitions. Then

$$\begin{aligned}
& \dim U \\
= \; & \dim\big(\text{null}(A^2 - A)\big) \\
= \; & \dim\big(\text{null}(D^2 - D)\big), \text{ since } X \in \text{null}(A^2 - A) \Leftrightarrow P^{-1}X \in \text{null}(D^2 - D) \\
= \; & \dim\big(\text{null}(\text{diag}(\lambda_1^2 - \lambda_1, \lambda_2^2 - \lambda_2, \ldots, \lambda_n^2 - \lambda_n))\big) \\
= \; & n - \text{rank}\big(\text{diag}(\lambda_1^2 - \lambda_1, \lambda_2^2 - \lambda_2, \ldots, \lambda_n^2 - \lambda_n)\big) \\
= \; & \#\text{ of zeros on the diagonal of } D^2 - D \\
= \; & \text{multiplicity of the eigenvalue 0 of } A \text{ plus the multiplicity of the eigenvalue 1 of } A \\
= \; & \dim E_0(A) + \dim E_1(A), \text{ since } A \text{ is diagonalizable}
\end{aligned}$$

Aside: This result is actually true for any $n \times n$ matrix, whether it is diagonalizable or not. But to prove this more general version of Question 9(b) you have to know the results of Exercise #24 in Section 4.1 and Exercise #27 in Section 4.3, which were not assigned. The argument, included here because some of you may have tried something similar, goes like this: for any vector $X \in U$, write $X = AX + (X - AX)$. Then

$$AX \in E_1(A), \text{ since } A(AX) = A^2X = AX,$$

and

$$X - AX \in E_0(A), \text{ since } A(X - AX) = AX - A^2X = AX - AX = 0.$$

Thus $U = E_0(A) + E_1(A)$. Moreover,

$$E_1(A) \cap E_0(A) = \{0\}, \text{ since } AX = X \text{ and } AX = 0 \text{ together imply } X = 0.$$

Then

$$\begin{aligned}
\dim U &= \dim\left(E_0(A) + E_1(A)\right) \\
&= \dim E_0(A) + \dim E_1(A) - \dim\left(E_0(A) \cap E_1(A)\right) \\
&= \dim E_0(A) + \dim E_1(A) - 0 \\
&= \dim E_0(A) + \dim E_1(A)
\end{aligned}$$

10. Find a basis for $U^\perp$ if

$$U = \operatorname{span}\left\{\begin{bmatrix} 1 & 0 & 1 & 3 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 0 & 2 & 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 & 0 & 1 & 1 \end{bmatrix}^T\right\}.$$

Solution: let $A = \begin{bmatrix} 1 & 0 & 1 & 3 & 1 \\ 1 & 0 & 2 & 0 & 1 \\ 1 & -1 & 0 & 1 & 1 \end{bmatrix}$; then $U^\perp = \text{null}(A)$. Find the basic solutions of $\text{null}(A)$:

$$\begin{aligned}
\text{null}(A) &= \text{null}\begin{bmatrix} 1 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 1 & 1 & 2 & 0 \end{bmatrix} \\
&= \text{null}\begin{bmatrix} 1 & 0 & 1 & 3 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix} \\
&= \text{null}\begin{bmatrix} 1 & 0 & 0 & 6 & 1 \\ 0 & 1 & 0 & 5 & 0 \\ 0 & 0 & 1 & -3 & 0 \end{bmatrix} \\
&= \operatorname{span}\left\{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}\right\}
\end{aligned}$$

So a basis for $U^\perp$ is

$$\left\{\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ -5 \\ 3 \\ 1 \\ 0 \end{bmatrix}\right\}.$$

11. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

Solution:

Step 1: Find the eigenvalues of A.

$$\begin{aligned}
\det(\lambda I - A) = \det \begin{bmatrix} \lambda-5 & 2 & -4 \\ 2 & \lambda-8 & -2 \\ -4 & -2 & \lambda-5 \end{bmatrix} &= \det \begin{bmatrix} \lambda-5 & 2 & -4 \\ 2 & \lambda-8 & -2 \\ 0 & 2\lambda-18 & \lambda-9 \end{bmatrix} \\
&= (\lambda-9)\det \begin{bmatrix} \lambda-5 & 2 & -4 \\ 2 & \lambda-8 & -2 \\ 0 & 2 & 1 \end{bmatrix} \\
&= (\lambda-9)\det \begin{bmatrix} \lambda-5 & 10 & 0 \\ 2 & \lambda-4 & 0 \\ 0 & 2 & 1 \end{bmatrix} \\
&= (\lambda-9)(\lambda^2-9\lambda) \\
&= \lambda(\lambda-9)^2
\end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 9$, repeated.

Step 2: Find mutually orthogonal eigenvectors of A.

$$E_9(A) = \text{null} \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} = \text{null} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

$$E_0(A) = \text{null} \begin{bmatrix} 5 & -2 & 4 \\ -2 & 8 & 2 \\ 4 & 2 & 5 \end{bmatrix} = \text{null} \begin{bmatrix} -1 & 4 & 1 \\ 5 & -2 & 4 \\ 4 & 2 & 5 \end{bmatrix} = \text{null} \begin{bmatrix} -1 & 4 & 1 \\ 0 & 18 & 9 \\ 0 & 18 & 9 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right\}.$$

OR: $E_0(A) = (E_9(A))^\perp$, which is the line passing through the origin, normal to the plane with equation $2x + y - 2z = 0$.

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} 2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ 1/3 & 0 & 4/\sqrt{18} \\ -2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

12. Let $U = \text{span}\left\{X_1 = [\ 0\ \ 1\ \ 0\ \ -1\ ]^T, X_2 = [\ 1\ \ 0\ \ 0\ \ 1\ ]^T, X_3 = [\ 0\ \ 0\ \ 1\ \ -1\ ]^T\right\}$; let $X = \left[\ 1\ \ 1\ \ 0\ \ 1\ \right]^T$. Find:

(a) [6 marks] an orthogonal basis of U.

Solution: Use the Gram-Schmidt algorithm.

$$F_1 = X_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}; F_2 = X_2 - \frac{X_2 \cdot F_1}{\|F_1\|^2} F_1 = X_2 + \frac{1}{2} F_1 = \frac{1}{2} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix};$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_3 \cdot F_2}{\|F_2\|^2} F_2 = X_3 - \frac{1}{2} F_1 + \frac{1}{6} F_2 = \frac{1}{3} \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

Optional: clear fractions and take

$$F_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, F_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, F_3 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ -1 \end{bmatrix}.$$

Either way $\{F_1, F_2, F_3\}$ is an orthogonal basis of U.

(b) [6 marks] $\text{proj}_U(X)$.

Solution: using $\{F_1, F_2, F_3\}$ with fractions cleared.

$$\begin{aligned} \text{proj}_U X &= \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \frac{X \cdot F_2}{\|F_2\|^2} F_2 + \frac{X \cdot F_3}{\|F_3\|^2} F_3 \\ &= \frac{0}{2} F_1 + \frac{2}{3} F_2 - \frac{1}{12} F_3 \\ &= \frac{1}{4} \begin{bmatrix} 5 \\ 3 \\ -1 \\ 3 \end{bmatrix}. \end{aligned}$$

Cross-check/Alternate Solution: $U^\perp = \text{span}\{Y\}$ with $Y = [\ -1\ \ 1\ \ 1\ \ 1\ ]^T$. Then

$$\text{proj}_U X = X - \text{proj}_{U^\perp}(X) = X - \frac{X \cdot Y}{\|Y\|^2} Y = X - \frac{1}{4} Y = \frac{1}{4} \begin{bmatrix} 5 \\ 3 \\ -1 \\ 3 \end{bmatrix}.$$

13. Find the least squares approximating quadratic for the data points

$$(-1,0),(0,2),(1,1),(1,-1).$$

Solution: Let the least squares approximating quadratic be $y = a + bx + cx^2$; let

$$M = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, Z = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, Y = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}.$$

The normal equations are:

$$M^TMZ = M^TY \Leftrightarrow \begin{bmatrix} 4 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

Approach 1: reduce the augmented matrix:

$$\left[\begin{array}{ccc|c} 4 & 1 & 3 & 2 \\ 1 & 3 & 1 & 0 \\ 3 & 1 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 3 & 1 & 3 & 0 \\ 1 & 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & -6 \\ 0 & 3 & 1 & -2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 3 & -6 \\ 0 & 0 & 8 & -16 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \end{array}\right].$$

Approach 2: use your favourite method to find the inverse of M^TM :

$$(M^TM)^{-1} = \frac{1}{8}\begin{bmatrix} 8 & 0 & -8 \\ 0 & 3 & -1 \\ -8 & -1 & 11 \end{bmatrix}; \text{ so } \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 8 & 0 & -8 \\ 0 & 3 & -1 \\ -8 & -1 & 11 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

Approach 3: use the adjoint formula for $(M^TM)^{-1}$ but simplify before you calculate:

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{\operatorname{adj}(M^TM)}{\det(M^TM)} = \frac{1}{8}\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} C_{11} \\ C_{12} \\ C_{13} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 0 \\ -8 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

Answer: the least squares approximating quadratic is

$$y = 2 - 2x^2.$$

Aside: note that the approximating quadratic goes halfway between the two data points $(1,1)$ and $(1,-1)$.