University of Toronto

Faculty of Applied Science and Engineering

Final Examination, April 2018

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1S - Linear Algebra

Examiner: D. Burbulla

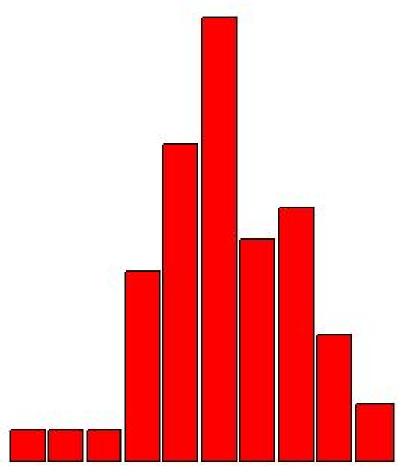
Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

- Six questions had passing averages; four had failing averages. On the whole, Questions 4, 8 and 10 were very poorly done, yet they were not complicated questions. Indeed, many students made questions 1(d), 4, 8 and 10 much more complicated and messy than necessary.
- Many students did not answer the question in Question 9; they attempted to find a linear or other least squares approximation instead. You must answer the question asked!

Breakdown of Results: 54 registered students wrote this test. The marks ranged from 4% to 97%, and the average was only 55.8%. Some statistics on grade distribution are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	3.7%
A	11.1%	80-89%	7.4%
B	14.8%	70-79%	14.8%
C	13.0%	60-69%	13.0%
D	25.9%	50-59%	25.9%
F	35.2%	40-49%	18.4%
		30-39%	11.1%
		20-29%	1.9%
		10-19%	1.9%
		0-9%	1.9%



1. [avg: 6.8/10] Fnd the following:

   (a) [2 marks] $\dim(S^\perp)$, if S is a subspace of $\mathbb{R}^7$ and $\dim(S) = 4$.

   Solution: $\dim(S^\perp) = 7 - \dim(S) = 7 - 4 = 3.$

   (b) [4 marks] $\det(-2A\,B^{-1}\,A^T)$, if A and B are 3×3 matrices, with $\det(A) = 3$ and $\det(B) = 5$.

   Solution: $\det(-2A\,B^{-1}\,A^T) = (-2)^3 \det(A)\det(B^{-1})\det(A^T) = (-8)(3)(5^{-1})(3) = -\dfrac{72}{5}$

   (c) [2 marks] the characteristic polynomial of $\begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}$.

   Solution: $\det \begin{bmatrix} x-4 & -3 \\ -6 & x-5 \end{bmatrix} = (x-4)(x-5) - 18 = x^2 - 9x + 2.$

   (d) [2 marks] $\det \begin{bmatrix} 4 & 3 & 10 & -3 \\ 13 & 5 & e & -5 \\ -9 & 4 & \pi & -4 \\ 14 & -1 & \sqrt{2} & 1 \end{bmatrix}$

   Solution: add the second column to the fourth column:

$$\det \begin{bmatrix} 4 & 3 & 10 & -3 \\ 13 & 5 & e & -5 \\ -9 & 4 & \pi & -4 \\ 14 & -1 & \sqrt{2} & 1 \end{bmatrix} = \det \begin{bmatrix} 4 & 3 & 10 & 0 \\ 13 & 5 & e & 0 \\ -9 & 4 & \pi & 0 \\ 14 & -1 & \sqrt{2} & 0 \end{bmatrix} = 0.$$

2. Let $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_4 = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$. Show $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ is an orthogonal set, and write $\vec{x}$ as a linear combination of $\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4$.

Solution: [avg: 9.1/10] show that each pair of vectors $\vec{u}_i, \vec{u}_j, i \neq j$ is orthogonal. There are six dot products you must calculate:

1. $\vec{u}_1 \cdot \vec{u}_2 = 2 + 0 - 2 + 0 = 0$
2. $\vec{u}_1 \cdot \vec{u}_3 = 0 - 0 - 1 + 1 = 0$
3. $\vec{u}_1 \cdot \vec{u}_4 = -2 + 0 - 1 + 3 = 0$
4. $\vec{u}_2 \cdot \vec{u}_3 = 0 - 2 + 2 + 0 = 0$
5. $\vec{u}_2 \cdot \vec{u}_4 = -4 + 2 + 2 + 0 = 0$
6. $\vec{u}_3 \cdot \vec{u}_4 = 0 - 4 + 1 + 3 = 0$

Now let $\vec{x} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4$. Using the results from Chapter 7 we can solve for c_i directly:

$$c_i = \frac{\vec{x} \cdot \vec{u}_i}{\|\vec{u}_i\|^2}.$$

Thus

$$c_1 = \frac{1 - 0 - 1 - 1}{3} = -\frac{1}{3}; c_2 = \frac{2 - 1 + 2 - 0}{9} = \frac{1}{3}; c_3 = \frac{0 + 2 + 1 - 1}{6} = \frac{1}{3}; c_4 = \frac{-2 - 2 + 1 - 3}{18} = -\frac{1}{3}.$$

OR you can solve a system of linear equations for c_i by reducing an augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 1 \\ 0 & 1 & -2 & 2 & -1 \\ -1 & 2 & 1 & 1 & 1 \\ 1 & 0 & 1 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 1 \\ 0 & 1 & -2 & 2 & -1 \\ 0 & 4 & 1 & -1 & 2 \\ 0 & -2 & 1 & 5 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 1 \\ 0 & 1 & -2 & 2 & -1 \\ 0 & 0 & 9 & -9 & 6 \\ 0 & 0 & -3 & 9 & -4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & -2 & 1 \\ 0 & 1 & -2 & 2 & -1 \\ 0 & 0 & 9 & -9 & 6 \\ 0 & 0 & 0 & 18 & -6 \end{array}\right];$$

and as before, $c_4 = -\frac{1}{3},\ c_3 = \frac{1}{3},\ c_2 = \frac{1}{3},\ c_1 = -\frac{1}{3}$. Either way

$$\vec{x} = -\frac{1}{3}\vec{u}_1 + \frac{1}{3}\vec{u}_2 + \frac{1}{3}\vec{u}_3 - \frac{1}{3}\vec{u}_4.$$

3. [avg: 6.9/10]

Let $A = \frac{1}{17}\begin{bmatrix} 8 & 15 \\ 15 & -8 \end{bmatrix}$. Find the eigenvalues of A and a basis for each eigenspace of A. Plot the eigenspaces of A in $\mathbb{R}^2$, and clearly indicate which eigenspace corresponds to which eigenvalue. Interpret your result geometrically.

Solution: A is a refection matrix, so the eigenvalues must be $\lambda = \pm 1$. OR: calculate them directly.

$$\det \begin{bmatrix} \lambda - 8/17 & -15/17 \\ -15/17 & \lambda + 8/17 \end{bmatrix} = \lambda^2 - \frac{64}{289} - \frac{225}{289} = \lambda^2 - 1 = (\lambda - 1)(\lambda + 1).$$

Eigenspaces: for $\lambda = 1$,

$$\text{null} \begin{bmatrix} 1 - 8/17 & -15/17 \\ -15/17 & 1 + 8/17 \end{bmatrix} = \text{null} \begin{bmatrix} 9 & -15 \\ -15 & 25 \end{bmatrix} = \text{null} \begin{bmatrix} 3 & -5 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\}.$$

For $\lambda = -1$,

$$\text{null} \begin{bmatrix} -1 - 8/17 & -15/17 \\ -15/17 & -1 + 8/17 \end{bmatrix} = \text{null} \begin{bmatrix} -25 & -15 \\ -15 & -9 \end{bmatrix} = \text{null} \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\}.$$

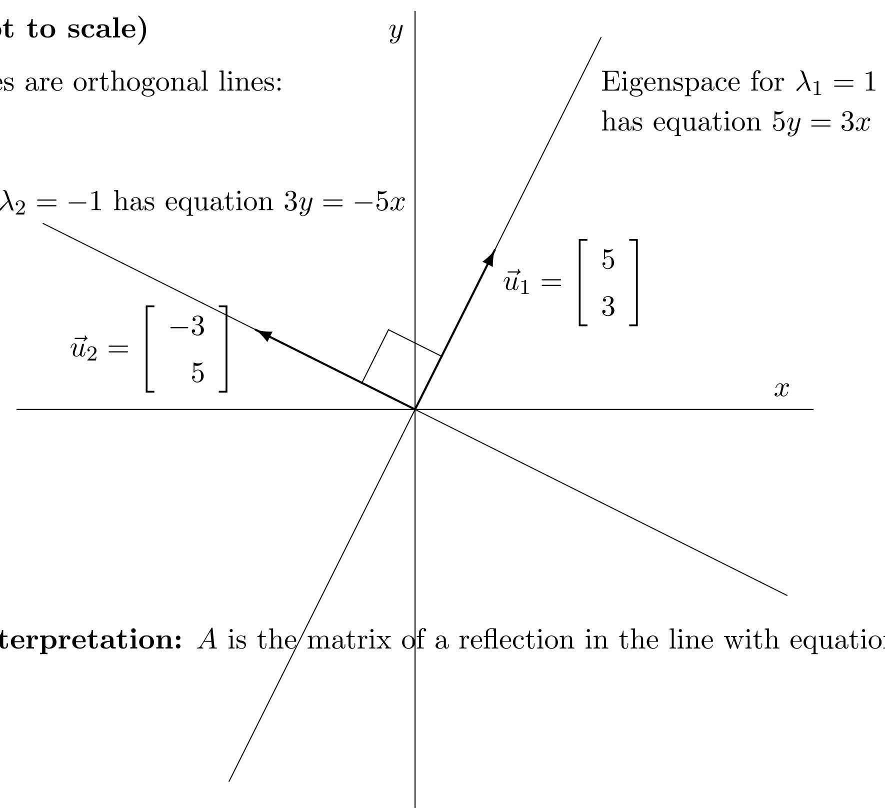
Thus

$$E_1(A) = \text{span} \left\{ \begin{bmatrix} 5 \\ 3 \end{bmatrix} \right\} \text{ and } E_{-1} = \text{span} \left\{ \begin{bmatrix} -3 \\ 5 \end{bmatrix} \right\},$$

which are lines with equations $y = \frac{3}{5}x$ and $y = -\frac{5}{3}x$, respectively.

Diagram: (not to scale)

The Eigenspaces are orthogonal lines:



Geometrical Interpretation: A is the matrix of a reflection in the line with equation $5y = 3x$.

4. [avg: 2.7/10] Let S be the set of vectors $\vec{x}$ in $\mathbb{R}^4$ such that

$$\det \begin{bmatrix} 1 & 0 & 1 & x_1 \\ 2 & 1 & 0 & x_2 \\ 3 & 1 & 1 & x_3 \\ 5 & 2 & -1 & x_4 \end{bmatrix} = 0.$$

(a) [5 marks] Explain why S is a subspace of $\mathbb{R}^4$.

Solution 1: take the cofactor expansion along the $4th$ column of the matrix to get

$$x_1 C_{14} + x_2 C_{24} + x_3 C_{34} + x_4 C_{44} = 0 \Leftrightarrow x_1 + x_2 - x_3 = 0.$$

Thus S is the set of the solutions to a single homogeneous equation, so S is a subspace of $\mathbb{R}^4$. Equivalently, $S = \text{null} \begin{bmatrix} C_{14} & C_{24} & C_{34} & C_{44} \end{bmatrix} = \text{null} \begin{bmatrix} -2 & -2 & 2 & 0 \end{bmatrix}$.

Solution 2: since the first three columns, $\vec{c}_1, \vec{c}_2, \vec{c}_3$, of the matrix are linearly independent, $\vec{x} \in S$ if and only if the columns of the matrix are linearly dependent if and only if $\vec{x} \in \text{span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$. Thus $S = \text{span}\{\vec{c}_1, \vec{c}_2, \vec{c}_3\}$, so S is a subspace of $\mathbb{R}^4$.

Solution 3: use the subspace test.

1. $\vec{0} \in S$ since any matrix with an all-zero column has determinant equal to 0.
2. if $\vec{x} \in S$ then $\det[\vec{c}_1\ \vec{c}_2\ \vec{c}_3\ k\vec{x}] = k \det[\vec{c}_1\ \vec{c}_2\ \vec{c}_3\ \vec{x}] = k \cdot 0 = 0$, so $k\,\vec{x} \in S$.
3. suppose $\vec{x}$ and $\vec{y}$ are both in S, then as in Solution 1,

$$\begin{aligned} \det[\vec{c}_1\ \vec{c}_2\ \vec{c}_3\ \vec{x}+\vec{y}] &= (x_1+y_1)C_{14} + (x_2+y_2)C_{24} + (x_3+y_3)C_{34} + (x_4+y_4)C_{44} \\ &= x_1C_{14} + x_2C_{24} + x_3C_{34} + x_4C_{44} + y_1C_{14} + y_2C_{24} + y_3C_{34} + y_4C_{44} \\ &= \det[\vec{c}_1\ \vec{c}_2\ \vec{c}_3\ \vec{x}] + \det[\vec{c}_1\ \vec{c}_2\ \vec{c}_3\ \vec{y}] \\ &= 0 + 0 = 0, \end{aligned}$$

so $\vec{x} + \vec{y} \in S$.

(b) [5 marks] Find a basis for S and its dimension.

Solution: from the second solution to part (a), a basis of S is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and $\dim(S) = 3$.

5. [avg: 7.6/10] Solve the system of differential equations

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} &= -x + 4y \\ \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} &= 8x - 5y \end{cases}$$

for x and y as functions of t, given that $(x, y) = (6, 12)$ when $t = 0$.

Solution: let $A = \begin{bmatrix} -1 & 4 \\ 8 & -5 \end{bmatrix}$; find the eigenvalues and eigenvectors of A.

Eigenvalues of A**:** $\det(\lambda I - A) = \det \begin{bmatrix} \lambda+1 & -4 \\ -8 & \lambda+5 \end{bmatrix} = \lambda^2 + 6\lambda - 27 = (\lambda+9)(\lambda-3)$. So the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -9$.

Eigenvectors of A**:**

$$\text{null}\begin{bmatrix} 3+1 & -4 \\ -8 & 3+5 \end{bmatrix} = \text{null}\begin{bmatrix} 4 & -4 \\ -8 & 8 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}; \text{ take } \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{null}\begin{bmatrix} -9+1 & -4 \\ --8 & -9+5 \end{bmatrix} = \text{null}\begin{bmatrix} -8 & -4 \\ -8 & -4 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}; \text{ take } \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The general solution to the system of differential equations is

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1\vec{v}_1 e^{4t} + c_2\vec{v}_2 e^{5t} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-9t}$$

To find c_1, c_2, let $t = 0$ and solve the system of linear equations

$$\begin{bmatrix} 6 \\ 12 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 12 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}.$$

Thus the solutions to the system of differential equations is

$$x = 8e^{3t} - 2e^{-9t} \text{ and } y = 8e^{3t} + 4e^{-9t}.$$

6. [avg: 6.5/10]

Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

Step 1: find the eigenvalues of A.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda & -1 & 1 \\ -1 & \lambda & 1 \\ 1 & 1 & \lambda \end{bmatrix} = \det \begin{bmatrix} \lambda & -1 & 1 \\ -1-\lambda & \lambda+1 & 0 \\ 0 & \lambda+1 & \lambda+1 \end{bmatrix}$$

$$= (\lambda+1)^2 \det \begin{bmatrix} \lambda & -1 & 1 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = (\lambda+1)^2(\lambda-2)$$

Thus the eigenvalues of A are $\lambda_1 = -1$, repeated, and $\lambda_2 = 2$.

Step 2: find an *orthogonal* basis of eigenvectors for each eigenspace.

$$\text{For } \lambda_1 = -1: \ \text{null} \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\};$$

$$\text{for } \lambda_2 = 2: \ \text{null} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 3 & 3 \\ -1 & 2 & 1 \\ 0 & 3 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Step 3: for the columns of P, take the unit, orthogonal eigenvectors and for the diagonal entries of D take the corresponding eigenvalues:

$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1/\sqrt{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

7. [avg: 4.5/10] Let $S = \text{span}\left\{ [\ 1\ \ 0\ \ 1\ \ 0\ ]^T, [\ 0\ \ 1\ \ 2\ \ 0\ ]^T, [\ 1\ \ 1\ \ 1\ \ -2\ ]^T \right\}$.

(a) [3 marks] Find a basis for $S^\perp$, the orthogonal complement of S.

Solution:

$$S^\perp = \text{null} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & -2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Take $\vec{w} = [\ 1\ \ 2\ \ -1\ \ 1\ ]^T$ as the basis for $S^\perp$.

(b) [4 marks] Find the standard matrix of the linear transformation $\text{perp}_S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$.

Solution 1: $\text{perp}_S(\vec{x}) = \text{proj}_{S^\perp}(\vec{x}) = \dfrac{\vec{x} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w} =$

$$\frac{(x_1 + 2x_2 - x_3 + x_4)}{7} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} x_1 + 2x_2 - x_3 + x_4 \\ 2x_1 + 4x_2 - 2x_3 + 2x_4 \\ -x_1 - 2x_2 + x_3 - x_4 \\ x_1 + 2x_2 - x_3 + x_4 \end{bmatrix},$$

from which you can read off the matrix: $[\text{perp}_S] = \dfrac{1}{7} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ -1 & -2 & 1 & -1 \\ 1 & 2 & -1 & 1 \end{bmatrix}$.

Solution 2: use definition of standard matrix.

$[\text{perp}_S] = [\text{perp}_S(\vec{e}_1)\ \text{perp}_S(\vec{e}_2)\ \text{perp}_S(\vec{e}_3)\ \text{perp}_S(\vec{e}_4)] = [\text{proj}_{\vec{w}}(\vec{e}_1)\ \text{proj}_{\vec{w}}(\vec{e}_2)\ \text{proj}_{\vec{w}}(\vec{e}_3)\ \text{proj}_{\vec{w}}(\vec{e}_4)]$

(c) [3 marks] What is the standard matrix of $\text{proj}_S : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$? (You can make use of part (b).)

Solution: use the fact that $[\text{proj}_S] = I - [\text{perp}_S]$. Thus

$$[\text{proj}_S] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -2 & 2 \\ -1 & -2 & 1 & -1 \\ 1 & 2 & -1 & 1 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 6 & -2 & 1 & -1 \\ -2 & 3 & 2 & -2 \\ 1 & 2 & 6 & 1 \\ -1 & -2 & -1 & 6 \end{bmatrix}.$$

8. [avg: 2.6/10] Prove the following:

(a) [5 marks] if A is an invertible $n \times n$ matrix with eigenvalue λ, then λ^{-1} is an eigenvalue of A^{-1}.

Proof: since λ is an eigenvalue of A there is a non-zero vector $\vec{v}$ such that $A\,\vec{v} = \lambda\,\vec{v}$; and since the matrix A is invertible, $\lambda \neq 0$. Then

$$\begin{aligned} A\,\vec{v} = \lambda\,\vec{v} \quad &\Rightarrow \quad \vec{v} = A^{-1}(\lambda\,\vec{v}) = \lambda\,A^{-1}\,\vec{v} \\ &\Rightarrow \quad \lambda^{-1}\,\vec{v} = A^{-1}\,\vec{v}. \end{aligned}$$

which means λ^{-1} is an eigenvalue of A^{-1} (with the same eigenvector $\vec{v}$..)

(b) [5 marks] if A is a 7×5 matrix with rank 4, then the matrix $A^T A$ is not invertible.

Proof: by the Rank-Nullity Theorem, the nullity of A is $5 - 4 = 1$. Thus $\dim(\text{null}(A)) = 1$, and there is a non-zero vector $\vec{x}$ such that $A\,\vec{x} = \vec{0}$. Then

$$\begin{aligned} A\,\vec{x} = \vec{0} \quad &\Rightarrow \quad A^T(A\,\vec{x}) = A^T\,\vec{0} \\ &\Rightarrow \quad (A^T A)\,\vec{x} = \vec{0}, \end{aligned}$$

Therefore the null space of $A^T A$ is also non-trivial, and so the matrix $A^T A$ is not invertible.

Continued...

9. [avg: 6.6/10] Use the method of least squares to find the best fitting quadratic function $f(x) = a + bx + cx^2$ for the five data points $(x, y) = (-2, 3),\ (-1, 2),\ (0, 0),\ (1, 2)$ and $(2, 8)$.

Solution: let $M = \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix}$; $\vec{y} = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 8 \end{bmatrix}$. The normal equations are $M^T M \begin{bmatrix} a \\ b \\ c \end{bmatrix} = M^T \vec{y}$.

Solve:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 4 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \\ 8 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 34 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 48 \end{bmatrix}$$

This system is easy to solve, since $b = 1$, by inspection. Then $5a + 10c = 15$ and $5a + 17c = 24$, from which you can deduce

$$7c = 9 \Leftrightarrow c = \frac{9}{7} \text{ and } a = 3 - 2c = 5 - \frac{18}{7} = \frac{3}{7}.$$

Thus the quadratic function that best fits the data has equation $y = \frac{3}{7} + x + \frac{9}{7}x^2$.

Continued...

10. [avg: 2.5/10] Find all linear transformations $L : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ such that

the range of L is span $\left\{ \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \right\}$ and the null space (or kernel) of L is span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

Solution 1: let A be the standard matrix of L. Then in terms of A,

$$(1),\ \operatorname{col}(A) = \operatorname{span} \left\{ \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \right\} \text{ and } (2),\ \operatorname{null}(A) = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

By (1),

$$A = \begin{bmatrix} -a & -b & -c \\ 5a & 5b & 5c \\ 2a & 2b & 2c \end{bmatrix},$$

for some scalars a, b, c. By (2),

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

which means

$$a + 2b + 3c = 0 \text{ and } a + b + c = 0.$$

Solving this system gives $a = c$ and $b = -2c$, for parameter c. Thus

$$A = \begin{bmatrix} -c & 2c & -c \\ 5c & -10c & 5c \\ 2c & -4c & 2c \end{bmatrix},$$

for $c \neq 0$. And

$$L(\vec{x}) = A\,\vec{x} = \begin{bmatrix} -c & 2c & -c \\ 5c & -10c & 5c \\ 2c & -4c & 2c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c \begin{bmatrix} -x_1 + 2x_2 - x_3 \\ 5x_1 - 10x_2 + 5x_3 \\ 2x_1 - 4x_2 + 2x_3 \end{bmatrix}$$

See the next page for an alternate solution.

This page is for rough work or for extra space to finish a previous problem. It will not be marked unless you have indicated in a previous question to look at this page.

Solution 2: extend the basis of the null space of L to a basis for $\mathbb{R}^3$ by including (say) the vector $\vec{e}_3$. Let A be the standard matrix of L. Then, for some non-zero scalar a,

$$A\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 5a \\ 0 & 0 & 2a \end{bmatrix} \Leftrightarrow A = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 5a \\ 0 & 0 & 2a \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 1 & 1 \end{bmatrix}^{-1}.$$

Use the Gaussian algorithm to find the inverse:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & -2 & 1 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right];$$

thus

$$A = \begin{bmatrix} 0 & 0 & -a \\ 0 & 0 & 5a \\ 0 & 0 & 2a \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 2 & -1 & 0 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} -a & 2a & -a \\ 5a & -10a & 5a \\ 2a & -4a & 2a \end{bmatrix}.$$

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The end.