University of Toronto

Faculty of Applied Science and Engineering

Solutions to Final Examination, April 2016

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT188H1S - Linear Algebra

Examiner: D. Burbulla

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

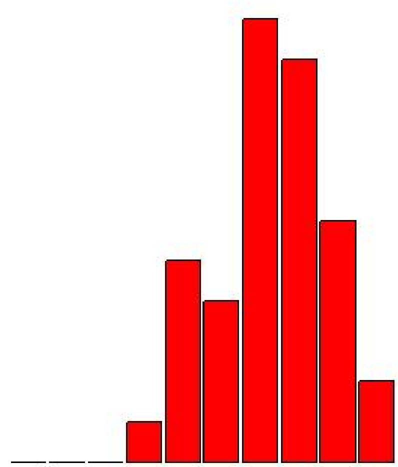
This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. The routine questions, 1, 2, 4(a), 5, 6 and 7 were, on the whole very well done.
2. The non-routine questions, 3, 4(b) and 8, were not well done; none of them had a passing average, even though 4(b) was very similar to December 2015 exam question 4, and 8(a) was actually quite simple.
3. Many students exhibited some very bad notation; this cost marks!

Breakdown of Results: all 39 students wrote this exam. The marks ranged from 37.5% to 93.75%, and the average was 67.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.1%
A	20.5%	80-89%	15.4%
B	25.6%	70-79%	25.6%
C	28.2%	60-69%	28.2%
D	10.3%	50-59%	10.3%
F	15.4%	40-49%	12.8%
		30-39%	2.6%
		20-29%	0.0 %
		10-19%	0.0%
		0-9%	0.0%



1.[avg: 9.2/10] Let

$$\mathbf{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \mathbf{u}_4 = \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix}.$$

Show $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an orthogonal set, and write $\mathbf{x}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$.

Solution: need to show $\mathbf{u}_i \cdot \mathbf{u}_j = 0$, for $i \neq j$. Since $\mathbf{u}_i \cdot \mathbf{u}_j = \mathbf{u}_j \cdot \mathbf{u}_i$, there are only six dot products you must check:

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = 1 + 0 - 1 + 0 = 0, \quad \mathbf{u}_1 \cdot \mathbf{u}_3 = 3 + 0 + 2 - 5 = 0, \quad \mathbf{u}_1 \cdot \mathbf{u}_4 = -2 + 0 + 3 - 1 = 0,$$

$$\mathbf{u}_2 \cdot \mathbf{u}_3 = 3 - 1 - 2 + 0 = 0, \quad \mathbf{u}_2 \cdot \mathbf{u}_4 = -2 + 5 - 3 + 0 = 0, \quad \mathbf{u}_3 \cdot \mathbf{u}_4 = -6 - 5 + 6 + 5 = 0.$$

Thus $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is an independent set of four vectors in $\mathbf{R}^4$, so it is a basis for $\mathbf{R}^4$, and

$$\begin{aligned} \mathbf{x} &= \frac{\mathbf{x} \cdot \mathbf{u}_1}{\|\mathbf{u}_1\|^2}\mathbf{u}_1 + \frac{\mathbf{x} \cdot \mathbf{u}_2}{\|\mathbf{u}_2\|^2}\mathbf{u}_2 + \frac{\mathbf{x} \cdot \mathbf{u}_3}{\|\mathbf{u}_3\|^2}\mathbf{u}_3 + \frac{\mathbf{x} \cdot \mathbf{u}_4}{\|\mathbf{u}_4\|^2}\mathbf{u}_4 \\ &= \frac{4+2-1}{1+1+1}\mathbf{u}_1 + \frac{4+3-2}{1+1+1}\mathbf{u}_2 + \frac{12-3+4+5}{9+1+4+25}\mathbf{u}_3 + \frac{-8+15+6+1}{4+25+9+1}\mathbf{u}_4 \\ &= \frac{5}{3}\mathbf{u}_1 + \frac{5}{3}\mathbf{u}_2 + \frac{6}{13}\mathbf{u}_3 + \frac{14}{39}\mathbf{u}_4 \end{aligned}$$

OR, do it the long way: solve the vector equation $\mathbf{x} = x_1\,\mathbf{u}_1 + x_2\,\mathbf{u}_2 + x_3\,\mathbf{u}_3 + x_4\,\mathbf{u}_4$ for x_1, x_2, x_3, x_4 by solving a linear system of equations. Using augmented matrices and good old row reduction:

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & -2 & 4 \\ 0 & 1 & -1 & 5 & 3 \\ 1 & -1 & 2 & 3 & 2 \\ -1 & 0 & 5 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 4 & -7 & 1 \\ 0 & 1 & -1 & 5 & 3 \\ 0 & -1 & 7 & 4 & 3 \\ 0 & 1 & 8 & -1 & 5 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 4 & -7 & 1 \\ 0 & 1 & -1 & 5 & 3 \\ 0 & 0 & 6 & 9 & 6 \\ 0 & 0 & 9 & -6 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 4 & -7 & 1 \\ 0 & 1 & -1 & 5 & 3 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 39 & 14 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 4 & -7 & 1 \\ 0 & 1 & -1 & 5 & 3 \\ 0 & 0 & 26 & 0 & 12 \\ 0 & 0 & 0 & 39 & 14 \end{array}\right],$$

from which

$$x_4 = \frac{14}{39},\ x_3 = \frac{6}{13},\ x_2 = \frac{5}{3},\ x_1 = \frac{5}{3},$$

as before.

2. [avg: 8.4/10]

Let $A = \frac{1}{5}\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$. Find the eigenvalues of A and a basis for each eigenspace of A. Plot the eigenspaces of A in $\mathbf{R}^2$, and clearly indicate which eigenspace corresponds to which eigenvalue.

Solution: be careful with the fractions!

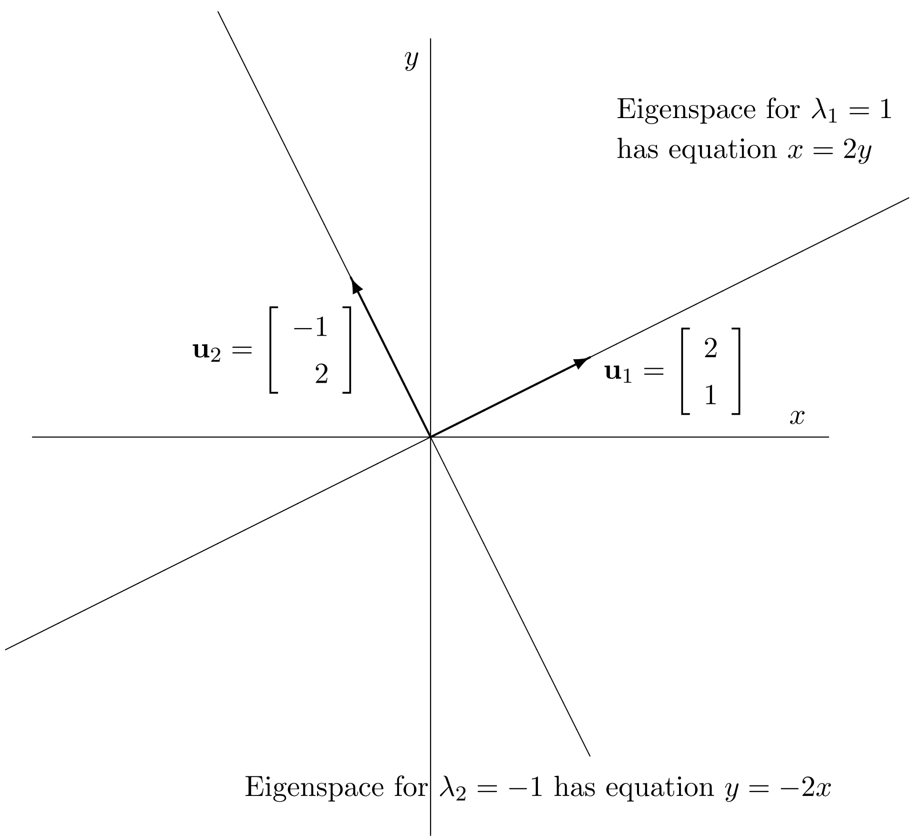
$$\det(\lambda I - A) = \det\begin{bmatrix} \lambda - 3/5 & -4/5 \\ -4/5 & \lambda + 3/5 \end{bmatrix} = \lambda^2 - \frac{9}{25} - \frac{16}{25} = \lambda^2 - 1,$$

so the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = -1$. Next, find a basis for each eigenspace:

$$\text{null}\begin{bmatrix} \lambda_1 - 3/5 & -4/5 \\ -4/5 & \lambda_1 + 3/5 \end{bmatrix} = \text{null}\begin{bmatrix} 2/5 & -4/5 \\ -4/5 & 8/5 \end{bmatrix} = \text{null}\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\};$$

$$\text{null}\begin{bmatrix} \lambda_2 - 3/5 & -4/5 \\ -4/5 & \lambda_2 + 3/5 \end{bmatrix} = \text{null}\begin{bmatrix} -8/5 & -4/5 \\ -4/5 & -2/5 \end{bmatrix} = \text{null}\begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} = \text{span}\left\{\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right\}.$$

Diagram:



The eigenvectors are orthogonal to each other; each eigenspace is a line passing through the origin.

3. [avg: 2.0/10] Let $\mathbf{v}$ be a unit column vector in $\mathbf{R}^n$; let $A = I - 2\,\mathbf{v}\,\mathbf{v}^T$, where I is the $n \times n$ identity matrix. Each of the following five statements is true. Give a *brief, clear* explanation why, for 2 marks each.

(a) $\mathbf{v}^T\,\mathbf{v} = 1$.

Solution: $\mathbf{v}^T\,\mathbf{v} = \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 1$, since $\mathbf{v}$ is a unit vector.

(b) $(\mathbf{v}\,\mathbf{v}^T)\,\mathbf{v} = \mathbf{v}$.

Solution: $(\mathbf{v}\,\mathbf{v}^T)\,\mathbf{v} = \mathbf{v}\,(\mathbf{v}^T\,\mathbf{v}) = \mathbf{v}(1) = \mathbf{v}$.

(c) $\mathbf{v}$ is an eigenvector of A, corresponding to eigenvalue $\lambda = -1$.

Solution: $A\,\mathbf{v} = (I - 2\,\mathbf{v}\,\mathbf{v}^T)\,\mathbf{v} = \mathbf{v} - 2(\mathbf{v}\,\mathbf{v}^T)\,\mathbf{v} = \mathbf{v} - 2\mathbf{v} = -\mathbf{v}$.

(d) A is symmetric.

Solution: $A^T = (I - 2\,\mathbf{v}\,\mathbf{v}^T)^T = I^T - 2\,(\mathbf{v}\,\mathbf{v}^T)^T = I - 2(\mathbf{v}^T)^T\,\mathbf{v}^T = I - 2\,\mathbf{v}\,\mathbf{v}^T = A$.

(e) A is orthogonal.

Solution: need to show $A^{-1} = A^T = A$, since A is symmetric.

$$\begin{aligned}
A\,A &= (I - 2\,\mathbf{v}\,\mathbf{v}^T)(I - 2\,\mathbf{v}\,\mathbf{v}^T) \\
&= I - 4\,\mathbf{v}\,\mathbf{v}^T + 4\,(\mathbf{v}\,\mathbf{v}^T)(\mathbf{v}\,\mathbf{v}^T) \\
&= I - 4\,\mathbf{v}\,\mathbf{v}^T + 4\,\mathbf{v}\,(\mathbf{v}^T\mathbf{v})\,\mathbf{v}^T) \\
&= I - 4\,\mathbf{v}\,\mathbf{v}^T + 4\,\mathbf{v}\,(1)\,\mathbf{v}^T \\
&= I - 4\,\mathbf{v}\,\mathbf{v}^T + 4\,\mathbf{v}\,\mathbf{v}^T \\
&= I.
\end{aligned}$$

Continued...

4.(a) [avg: 3.2/4] Find

$$\det \begin{bmatrix} 1 & 0 & -2 & 4 \\ 2 & 1 & 5 & -2 \\ -1 & 3 & 2 & -1 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

Solution: use appropriate row and/or column operations, and properties of determinants.

$$\begin{aligned}
\det \begin{bmatrix} 1 & 0 & -2 & 4 \\ 2 & 1 & 5 & -2 \\ -1 & 3 & 2 & -1 \\ 1 & 2 & 1 & 1 \end{bmatrix} &= \det \begin{bmatrix} 1 & 0 & -2 & 4 \\ 2 & 1 & 5 & -2 \\ -7 & 0 & -13 & 5 \\ -3 & 0 & -9 & 5 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & -2 & 4 \\ -7 & -13 & 5 \\ -3 & -9 & 5 \end{bmatrix} \\
&= \det \begin{bmatrix} 1 & -2 & 4 \\ 0 & -27 & 33 \\ 0 & -15 & 17 \end{bmatrix} \\
&= \det \begin{bmatrix} -27 & 33 \\ -15 & 17 \end{bmatrix} \\
&= (-3)(3) \det \begin{bmatrix} 3 & 11 \\ 5 & 17 \end{bmatrix} \\
&= (-9)(51 - 55) \\
&= (-9)(-4) \\
&= 36
\end{aligned}$$

Continued...

4.(b) [avg: 2.9/6] Find all matrices A such that $A^2 = 0$ and

$$\operatorname{col}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\}.$$

Solution:

$$\operatorname{col}(A) = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right\} \Rightarrow A = \begin{bmatrix} a & b & c \\ a & b & c \\ -a & -b & -c \end{bmatrix}$$

for some scalars a, b, c. Since $A^2 = AA = 0$, we have that $\operatorname{col}(A) \subset \operatorname{null}(A)$; that is,

$$\begin{bmatrix} a & b & c \\ a & b & c \\ -a & -b & -c \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow a + b - c = 0 \Leftrightarrow c = a + b.$$

Thus

$$A = \begin{bmatrix} a & b & a+b \\ a & b & a+b \\ -a & -b & -a-b \end{bmatrix}. \quad (*)$$

Now use the given fact that $A^2 = 0$ to find conditions on a, b :

$$\begin{aligned} A^2 &= \begin{bmatrix} a & b & a+b \\ a & b & a+b \\ -a & -b & -a-b \end{bmatrix}\begin{bmatrix} a & b & a+b \\ a & b & a+b \\ -a & -b & -a-b \end{bmatrix} \\ &= \begin{bmatrix} a^2 + ab - a(a+b) & ab + b^2 - b(a+b) & a(a+b) + b(a+b) - (a+b)^2 \\ a^2 + ab - a(a+b) & ab + b^2 - b(a+b) & a(a+b) + b(a+b) - (a+b)^2 \\ -a^2 - ab + a(a+b) & -ab - b^2 + b(a+b) & -a(a+b) - b(a+b) + (a+b)^2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

So no further conditions on a or b are required; they can be any values, as long as not both a and b are zero: A is then as given in (*).

5. [avg: 8.8/10] Find the solution to the system of linear differential equations $\begin{cases} y_1' &= 6y_1 + 2y_2 \\ y_2' &= y_1 + 5y_2 \end{cases}$, where y_1, y_2 are functions of t, and $y_1(0) = 4,\ y_2(0) = -1$.

Solution: let the coefficient matrix be $A = \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

$$\det(\lambda I - A) = (\lambda - 6)(\lambda - 5) - 2 = \lambda^2 - 11\lambda + 28 = (\lambda - 7)(\lambda - 4) = 0 \Rightarrow \lambda = 7 \text{ or } 4.$$

Find an eigenvector for the eigenvalue $\lambda_1 = 7$:

$$(\lambda_1 I - A|\mathbf{0}) = \left[\begin{array}{cc|c} 1 & -2 & 0 \\ -1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{take } \mathbf{u_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Find an eigenvector for the eigenvalue $\lambda_2 = 4$:

$$(\lambda_2 I - A|\mathbf{0}) = \left[\begin{array}{cc|c} -2 & -2 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{take } \mathbf{u_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then the general solution to the system of differential equations is

$$\mathbf{y} = c_1 \mathbf{u}_1 e^{7t} + c_2 \mathbf{u}_2 e^{4t}.$$

To find the constants c_1, c_2, use the initial conditions, with $t = 0$:

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{7t} - 2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$$

and

$$y_1 = 2\,e^{7t} + 2\,e^{4t};\ y_2 = e^{7t} - 2\,e^{4t}.$$

Continued...

6. [avg: 7.5/10] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\det \begin{bmatrix} \lambda-4 & -2 & 2 \\ -2 & \lambda-1 & 1 \\ 2 & 1 & \lambda-1 \end{bmatrix} = \det \begin{bmatrix} \lambda-4 & -2 & 0 \\ -2 & \lambda-1 & \lambda \\ 2 & 1 & \lambda \end{bmatrix} = \lambda \det \begin{bmatrix} \lambda-4 & -2 & 0 \\ -2 & \lambda-1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \lambda \det \begin{bmatrix} \lambda-4 & -2 & 0 \\ -4 & \lambda-2 & 0 \\ 2 & 1 & 1 \end{bmatrix} = \lambda \det \begin{bmatrix} \lambda-4 & -2 \\ -4 & \lambda-2 \end{bmatrix}$$

$$= \lambda(\lambda^2 - 6\lambda + 8 - 8) = \lambda^2 (\lambda - 6)$$

So the eigenvalues of A are $\lambda_1 = 0$, repeated; and $\lambda_2 = 6$.

Step 2: recall null$(A - \lambda I)$ is the eigenspace of A corresponding to the eigenvalue λ.
Find three mutually **orthogonal** eigenvectors of A. For $\lambda_1 = 0$,

$$\text{null}(A) = \text{null} \begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\};$$

and for $\lambda_2 = 6$,

$$\text{null}(6\,I - A) = \text{null} \begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix}; \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Continued...

7. [avg: 8.2/10] Let $S = \text{span}\left\{\begin{bmatrix} 2 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 0 & 2 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

Solution: call the three given vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and apply the Gram-Schmidt algorithm to find an orthogonal basis $\mathbf{f}_2, \mathbf{f}_2, \mathbf{f}_3$. Take $\mathbf{f}_1 = \mathbf{x}_1$,

$$\mathbf{f}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 5 \end{bmatrix},$$

$$\mathbf{f}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 2 & -3 & 3 & 4 \end{bmatrix}^T$. Find $\text{proj}_S(\mathbf{x})$.

Solution: use the projection formula, and your orthogonal basis from part (a).

$$\text{proj}_S\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 = \frac{1}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{2}{5}\begin{bmatrix} -1 \\ 2 \\ 0 \\ 5 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix}.$$

Alternate Solution: observe that $S^{\perp} = \text{span}\{\mathbf{y}\}$ with $\mathbf{y} = \begin{bmatrix} 1 & -2 & 0 & 1 \end{bmatrix}^T$. Then

$$\text{proj}_S\mathbf{x} = \mathbf{x} - \text{proj}_{S^\perp}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} = \mathbf{x} - \frac{12}{6}\mathbf{y} = \mathbf{x} - 2\mathbf{y} = \begin{bmatrix} 0 & 1 & 3 & 2 \end{bmatrix}^T.$$

8. [avg: 3.8/10]

Let $A = \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$. (a) [5 marks] Show that $\mathbf{u} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ is an eigenvector of A. What is the corresponding eigenvalue?

Solution: compute $A\mathbf{u}$, it should be a multiple of $\mathbf{u}$:

$$\begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+c+b \\ b+a+c \\ c+b+a \end{bmatrix} = (a+b+c) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

So the corresponding eigenvalue of A is $\lambda = a+b+c$.

(b) [5 marks] Show that the eigenvalue of A that you found in part (a) is the only real eigenvalue of A, if $b \neq c$. BONUS: what happens if $b = c$?

Solution: this is an exercise in calculating the determinant algebraically, as opposed to numerically.

$$\begin{aligned}
\det(\lambda I - A) = \det \begin{bmatrix} \lambda - a & -c & -b \\ -b & \lambda - a & -c \\ -c & -b & \lambda - a \end{bmatrix} &= \det \begin{bmatrix} \lambda - a - b - c & -c + \lambda - a - b & -b - c + \lambda - a \\ -b & \lambda - a & -c \\ -c & -b & \lambda - a \end{bmatrix} \\
&= (\lambda - a - b - c) \det \begin{bmatrix} 1 & 1 & 1 \\ -b & \lambda - a & -c \\ -c & -b & \lambda - a \end{bmatrix} \\
&= (\lambda - a - b - c) \det \begin{bmatrix} 1 & 0 & 0 \\ -b & \lambda - a + b & -c + b \\ -c & -b + c & \lambda - a + c \end{bmatrix} \\
&= (\lambda - a - b - c) \det \begin{bmatrix} \lambda - a + b & -c + b \\ -b + c & \lambda - a + c \end{bmatrix} \\
&= (\lambda - a - b - c)((\lambda - a)^2 + (b+c)(\lambda - a) + bc + (b-c)^2)
\end{aligned}$$

Thus $\lambda = a+b+c$, or

$$\lambda - a = \frac{-(b+c) \pm \sqrt{(b+c)^2 - 4bc - 4(b-c)^2}}{2} = \frac{-(b+c) \pm \sqrt{-3(b-c)^2}}{2}.$$

If $b \neq c$, the only real eigenvalue is $\lambda = a+b+c$. BONUS: but if $b = c$, then A is symmetric and its real eigenvalues are

$$\lambda = a+b+c \text{ and } \lambda = a - b, \text{ repeated.}$$

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