University of Toronto

Faculty of Applied Science and Engineering

Final Examination, April 2015

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

Solutions for **MAT188H1S - Linear Algebra**

Examiner: D. Burbulla

Exam Type: A. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

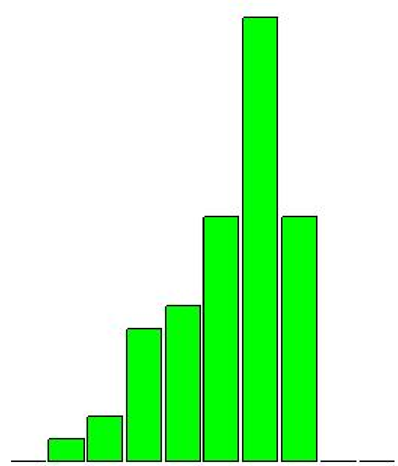
General Comments:

1. Finding the eigenvalues in Question 6 caused many students problems.

2. Questions 4 and 8 were total mysteries to most students; yet 4(a) and 4(b) were straightforward, and 8(a) was a WeBWorK question. On the other hand, 4(c) and 8(b) were *supposed* to be challenging.

Breakdown of Results: all 58 students wrote this exam. The marks ranged from 18.75% to 76.25%, and the average was 55.82%. (I will add 3.4 marks to everybody's exam score to make the average on the exam 60.01%) Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0%
A	0.0%	80-89%	0.0%
B	18.97%	70-79%	18.97%
C	34.48%	60-69%	34.48%
D	18.97%	50-59%	18.97%
F	27.58%	40-49%	12.07%
		30-39%	10.34%
		20-29%	3.45%
		10-19%	1.72%
		0-9%	0.0%



PART I : No explanation is necessary.

1. [avg: 4.7/10] Big Theorem, Final Exam Version: Let $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$ be a set of n vectors in $\mathbb{R}^n$, let

$$A = \begin{bmatrix} \mathbf{a_1} & \mathbf{a_2} & \ldots & \mathbf{a}_n \end{bmatrix}$$

be the matrix with the vectors in $\mathcal{A}$ as its columns, and let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear transformation defined by $T(\mathbf{x}) = A\,\mathbf{x}$. Decide if the following statements are equivalent to the statement, "A is invertible." Circle Yes if the statement is equivalent to "A is invertible," and No if it isn't.

Note: +1 for each correct choice; −1 for each incorrect choice; and 0 for each part left blank.

(a) $\det(A) \neq 0$. **[Yes]** No

(b) A is diagonalizable. Yes **[No]**

(c) The reduced echelon form of A is I, the $n \times n$ identity matrix. **[Yes]** No

(d) T is one-to-one. **[Yes]** No

(e) $\mathbf{0}$ is not in $\text{row}(A)$. Yes **[No]**

(f) $\text{range}(T) = \mathbb{R}^n$. **[Yes]** No

(g) $\text{row}(A) = \text{span}(\mathcal{A})$. Yes **[No]**

(h) $\dim(\text{col}(A)) + \dim(\text{null}(A)) = n$. Yes **[No]**

(i) $A = A^T$. Yes **[No]**

(j) $\lambda = 0$ is not an eigenvalue of A. **[Yes]** No

PART II : Present **COMPLETE** solutions to the following questions in the space provided.

2. [avg: 8.1/10] Find the following:

   (a) [2 marks] $\dim(S^{\perp})$, if S is a subspace of $\mathbb{R}^7$ and $\dim(S) = 3$.

   Solution: $\dim(S^{\perp}) = 7 - \dim(S) = 7 - 3 = 4$.

   (b) [2 marks] $\det(-3A^T B^2)$, if A and B are 3×3 matrices with $\det(A) = 1$ and $\det(B) = 2$.

   Solution: $\det(-3A^T B^2) = (-3)^3 \det(A)\,(\det(B))^2 = -27\,(1)(2^2) = -108$.

   (c) [2 marks] $\text{proj}_{\mathbf{a}}\mathbf{u}$, if $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{a} = \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}$.

   Solution: $\text{proj}_{\mathbf{a}}\mathbf{u} = \dfrac{\mathbf{u} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\,\mathbf{a} = \dfrac{5}{49} \begin{bmatrix} 6 \\ -3 \\ 2 \end{bmatrix}$.

   (d) [2 marks] $\begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}^{-1}$.

   Solution: $\begin{bmatrix} 4 & 3 \\ 6 & 5 \end{bmatrix}^{-1} = \dfrac{1}{20 - 18} \begin{bmatrix} 5 & -3 \\ -6 & 4 \end{bmatrix} = \dfrac{1}{2} \begin{bmatrix} 5 & -3 \\ -6 & 4 \end{bmatrix}$

   (e) [2 marks] $\det \begin{bmatrix} 4 & 3 & 10 & 6 \\ 13 & 5 & e & 10 \\ -9 & 4 & \pi & 8 \\ 14 & -1 & \sqrt{2} & -2 \end{bmatrix}$.

   Soluton: add $-2C_2$ to C_4 to get $\det \begin{bmatrix} 4 & 3 & 10 & 6 \\ 13 & 5 & e & 10 \\ -9 & 4 & \pi & 8 \\ 14 & -1 & \sqrt{2} & -2 \end{bmatrix} = \det \begin{bmatrix} 4 & 3 & 10 & 0 \\ 13 & 5 & e & 0 \\ -9 & 4 & \pi & 0 \\ 14 & -1 & \sqrt{2} & 0 \end{bmatrix} = 0$.

3. [avg: 6.7/10] For any real numbers a, b, c, let $A = \begin{bmatrix} 1 & a & b \\ -a & 1 & c \\ -b & -c & 1 \end{bmatrix}$ and let $B = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

(a) [6 marks] Determine for which values of a, b, c, if any, A and B are invertible.

Soluton: use the formula for the determinant of a 3×3 matrix.

$$\det(A) = 1 - abc + abc + c^2 + b^2 + a^2 = 1 + a^2 + b^2 + c^2 \geq 1$$

and

$$\det(B) = 0 - abc + abc + 0 + 0 + 0 = 0.$$

Thus

- A is invertible for all values of a, b, c.
- B is not invertible for any values of a, b, c.

(b) [4 marks] If either A or B is invertible, find its inverse.

Soluton: use the adjoint formula.

$$\begin{aligned} A^{-1} &= \frac{1}{\det(A)} \operatorname{adj}(A) \\ &= \frac{1}{1+a^2+b^2+c^2} \begin{bmatrix} 1+c^2 & -(-a+bc) & ac+b \\ -(a+bc) & 1+b^2 & -(-c+ab) \\ ac-b & -(c+ab) & 1+a^2 \end{bmatrix}^T \\ &= \frac{1}{1+a^2+b^2+c^2} \begin{bmatrix} 1+c^2 & -a-bc & ac-b \\ a-bc & 1+b^2 & -c-ab \\ ac+b & c-ab & 1+a^2 \end{bmatrix} \end{aligned}$$

4. [avg: 2.1/10] Suppose A is an $n \times n$ matrix such that $A^2 = 5A$.

(a) [4 marks] What are the possible eigenvalues of A?

Solution: let λ be an eigenvalue of A with corresponding eigenvector $\mathbf{v}$. Then

$$A\mathbf{v} = \lambda\mathbf{v} \Rightarrow A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda A\mathbf{v} = \lambda^2\mathbf{v}.$$

Thus

$$A^2\mathbf{v} = (5A)\mathbf{v} \Rightarrow \lambda^2\mathbf{v} = 5\lambda\mathbf{v} \Rightarrow \lambda^2 = 5\lambda,$$

since $\mathbf{v} \neq \mathbf{0}$. So the only possible eigenvalues of A are $\lambda = 0$ or $\lambda = 5$.

(b) [2 marks] What must A be if it is invertible?

Solution: if A is invertible then $A^2 = 5A \Rightarrow A^2A^{-1} = 5AA^{-1} \Rightarrow A = 5I$.

(c) [4 marks] Suppose that A is diagonalizable. Find the rank and nullity of A in terms of the multiplicities of its eigenvalues.

Solution: let the multiplicity of $\lambda = 0$ be m_0; let the multiplicity of $\lambda = 5$ be m_5. Then the characteristic polynomial of A is $x^{m_0}(x-5)^{m_5}$ and $m_0 + m_5 = n$. Let $E_\lambda(A)$ be the eigenspace of A corresponding to the eigenvalue λ of A. Since A is diagonalizable we have

$$\dim(E_0(A)) = m_0 \text{ and } \dim(E_5(A)) = m_5.$$

- Since $E_0(A) = \text{null}(A)$, we have that the nullity of A is m_0.
- Now we have

$$\text{nullity}(A) + \text{rank}(A) = n, \ \text{nullity}(A) = m_0, \text{ and } m_0 + m_5 = n,$$

from which it follows that the rank of A is m_5.

Note: everything above makes sense in the two extreme cases: $m_0 = 0$ and $m_5 = n$, or $m_0 = n$ and $m_5 = 0$. In the first case, $A = 5I$; in the second case, $A = 0$, the zero matrix.

5. [avg: 8.0/10] Find the solution to the system of linear differential equations

$$\begin{aligned} y_1' &= -\frac{3}{2}y_1 + \frac{1}{2}y_2 \\ y_2' &= y_1 - y_2 \end{aligned}$$

where y_1, y_2 are functions of t, and $y_1(0) = 5,\ y_2(0) = 4$.

Solution: let the coefficient matrix be $A = \begin{bmatrix} -3/2 & 1/2 \\ 1 & -1 \end{bmatrix}$. Find the eigenvalues and eigenvectors of A.

$$\det(\lambda I - A) = (\lambda + 3/2)(\lambda + 1) - 1/2 = \lambda^2 + \frac{5}{2}\lambda + 1 = (\lambda + 2)(\lambda + 1/2) = 0 \Rightarrow \lambda = -2 \text{ or } -1/2.$$

For the eigenvalue $\lambda_1 = -2$:

$$(\lambda_1 I_2 - A|\mathbf{0}) = \left[\begin{array}{cc|c} -1/2 & -1/2 & 0 \\ -1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{so take } \mathbf{u_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For the eigenvalue $\lambda_2 = -1/2$:

$$(\lambda_2 I_2 - A|\mathbf{0}) = \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ -1 & 1/2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right];\ \text{so take } \mathbf{u_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then the general solution to the system of differential equations is

$$\mathbf{y} = c_1\mathbf{u}_1 e^{-2t} + c_2\mathbf{u}_2 e^{-t/2}.$$

To find c_1, c_2 use the initial conditions, with $t = 0$:

$$\begin{bmatrix} 5 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-2t} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-t/2}$$

and

$$y_1 = 2e^{-2t} + 3e^{-t/2};\ y_2 = -2e^{-2t} + 6e^{-t/2}.$$

6. [avg: 5.9/10] Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}.$$

Step 1: Find the eigenvalues of A.

$$\det \begin{bmatrix} \lambda-3 & 2 & -4 \\ 2 & \lambda-6 & -2 \\ -4 & -2 & \lambda-3 \end{bmatrix} = \det \begin{bmatrix} \lambda-3 & 2 & -4 \\ 2 & \lambda-6 & -2 \\ 0 & 2\lambda-14 & \lambda-7 \end{bmatrix} = (\lambda-7)\det \begin{bmatrix} \lambda-3 & 2 & -4 \\ 2 & \lambda-6 & -2 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= (\lambda-7)\det \begin{bmatrix} \lambda-3 & 10 & -4 \\ 2 & \lambda-2 & -2 \\ 0 & 0 & 1 \end{bmatrix} = (\lambda-7)\det \begin{bmatrix} \lambda-3 & 10 \\ 2 & \lambda-2 \end{bmatrix}$$

$$= (\lambda-7)(\lambda^2-5\lambda-14) = (\lambda-7)(\lambda-7)(\lambda+2) = (\lambda-7)^2(\lambda+2)$$

So the eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = 7$, repeated.

Step 2: let $E_\lambda(A) = \text{null}(\lambda I - A)$ be the eigenspace of A corresponding to the eigenvalue λ. Find three mutually **orthogonal** eigenvectors of A. $E_{-2}(A) =$

$$\text{null} \begin{bmatrix} -5 & 2 & -4 \\ 2 & -8 & -2 \\ -4 & -2 & -5 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -4 & -1 \\ -5 & 2 & -4 \\ -4 & -2 & -5 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -4 & -1 \\ 0 & -18 & -9 \\ 0 & -18 & -9 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$$\text{and } E_7(A) = \text{null} \begin{bmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{bmatrix} = \text{null} \begin{bmatrix} 2 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix} \right\}.$$

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} -2/3 & 1/\sqrt{2} & -1/\sqrt{18} \\ -1/3 & 0 & 4/\sqrt{18} \\ 2/3 & 1/\sqrt{2} & 1/\sqrt{18} \end{bmatrix}; \quad D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

7. [avg: 7.2/10] Let $S = \text{span}\left\{[\ 1\quad 1\quad 0\quad 1\ ]^T, [\ 0\quad 1\quad 1\quad 1\ ]^T, [\ 1\quad 0\quad 1\quad 1\ ]^T\right\}$.

(a) [5 marks] Find an orthogonal basis of S.

Solution: call the three given vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ and apply the Gram-Schmidt algorithm to find an orthogonal basis $\mathbf{f}_2, \mathbf{f}_2, \mathbf{f}_3$. Take $\mathbf{f}_1 = \mathbf{x}_1$,

$$\mathbf{f}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix},$$

$$\mathbf{f}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{15}\begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \\ 3 \\ 1 \end{bmatrix}.$$

Then $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ is an orthogonal basis of S.

(b) [5 marks] Let $\mathbf{x} = \begin{bmatrix} 1 & 5 & 3 & 1 \end{bmatrix}^T$. Find $\text{proj}_S(\mathbf{x})$.

Solution: use the projection formula.

$$\text{proj}_S\mathbf{x} = \frac{\mathbf{x} \cdot \mathbf{f}_1}{\|\mathbf{f}_1\|^2}\mathbf{f}_1 + \frac{\mathbf{x} \cdot \mathbf{f}_2}{\|\mathbf{f}_2\|^2}\mathbf{f}_2 + \frac{\mathbf{x} \cdot \mathbf{f}_3}{\|\mathbf{f}_3\|^2}\mathbf{f}_3 = \frac{7}{3}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + \frac{13}{15}\begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} - \frac{7}{35}\begin{bmatrix} 3 \\ -4 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 2 \\ 3 \end{bmatrix}.$$

Alternate Solution: $S^\perp = \text{span}\{\mathbf{y}\}$ with $\mathbf{y} = [\ 1\quad 1\quad 1\quad -2\ ]^T$. Then

$$\text{proj}_S\mathbf{x} = \mathbf{x} - \text{proj}_{S^\perp}(\mathbf{x}) = \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{y}\|^2}\mathbf{y} = \mathbf{x} - \frac{7}{7}\mathbf{y} = [\ 0\quad 4\quad 2\quad 3\ ]^T.$$

8. [avg: 1.9/10] The parts of this question are unrelated. Each part is worth 5 marks.

(a) Suppose $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$ are four vectors in $\mathbb{R}^4$ and $\det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4\ ] = 5$. Find the value of $\det[\ \mathbf{a}_1 \quad 3\mathbf{a}_2 + 6\mathbf{a_4} \quad \mathbf{a}_3 \quad 2\mathbf{a}_4 - 8\mathbf{a}_2\ ]$.

Solution:

$$\begin{aligned}
\det[\ \mathbf{a}_1 \quad 3\mathbf{a}_2 + 6\mathbf{a_4} \quad \mathbf{a}_3 \quad 2\mathbf{a}_4 - 8\mathbf{a}_2\ ] &= (3)(2)\det[\ \mathbf{a}_1 \quad \mathbf{a}_2 + 2\mathbf{a_4} \quad \mathbf{a}_3 \quad \mathbf{a}_4 - 4\mathbf{a}_2\ ] \\
&= 6\det[\ \mathbf{a}_1 \quad \mathbf{a}_2 + 2\mathbf{a_4} \quad \mathbf{a}_3 \quad \mathbf{a}_4 - 4\mathbf{a}_2 + 4(\mathbf{a}_2 + 2\mathbf{a_4})\ ] \\
&= 6\det[\ \mathbf{a}_1 \quad \mathbf{a}_2 + 2\mathbf{a_4} \quad \mathbf{a}_3 \quad 9\mathbf{a}_4\ ] \\
&= 54\det[\ \mathbf{a}_1 \quad \mathbf{a}_2 + 2\mathbf{a_4} \quad \mathbf{a}_3 \quad \mathbf{a}_4\ ] \\
&= 54\det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{a}_4\ ] \\
&= 54(5) = 270
\end{aligned}$$

(b) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ be three linearly independent vectors in $\mathbb{R}^4$. Define $T : \mathbb{R}^4 \longrightarrow \mathbb{R}$ by $T(\mathbf{x}) = \det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{x}\ ]$. Show that T is a linear transformation and find range(T).

Solution: need to show (1) $T(k\mathbf{x}) = kT(\mathbf{x})$ and (2) $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$:

1. $T(k\mathbf{x}) = \det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad k\mathbf{x}\ ] = k\det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{x}\ ] = kT(\mathbf{x})$.
2.

$$\begin{aligned}
T(\mathbf{x}+\mathbf{y}) &= \det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{x}+\mathbf{y}\ ] \\
&= (x_1+y_1)C_{14} + (x_2+y_2)C_{24} + (x_3+y_3)C_{34} + (x_4+y_4)C_{44} \\
&= x_1C_{14} + x_2C_{24} + x_3C_{34} + x_4C_{44} + y_1C_{14} + y_2C_{24} + y_3C_{34} + y_4C_{44} \\
&= \det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{x}\ ] + \det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{y}\ ] \\
&= T(\mathbf{x}) + T(\mathbf{y})
\end{aligned}$$

Finally, the range of T is either $\{0\}$ or all of $\mathbb{R}$. In the former case, $\dim(\ker(T)) = 4$; in the latter case, $\dim(\ker(T)) = 3$. But $\dim(\ker(T)) = 3$, since

$$T(\mathbf{x}) = 0 \Leftrightarrow \det[\ \mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{x}\ ] = 0 \Leftrightarrow \mathbf{x} \in \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\},$$

since the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are given to be linearly independent. So

$$\ker(T) = \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\};$$

and it must be the case that range$(T) = \mathbb{R}$.

This page is for rough work; it will only be marked if you indicate you want it marked.