University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING
Solutions to **FINAL EXAMINATION, APRIL, 2010**
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MSE

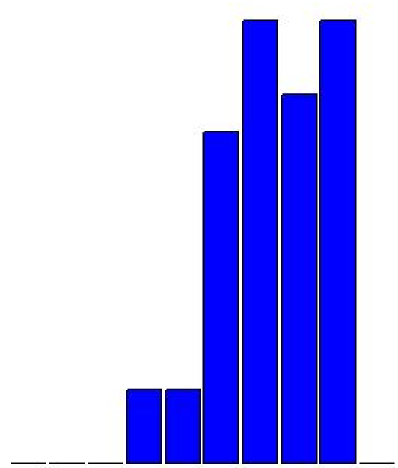
MAT188H1S - LINEAR ALGEBRA
Exam Type: A

General Comments:

1. Most of this exam was very similar to the exam of Dec 2009: in Questions 6, 8, 11, 12 and 13 only the numbers were changed; Question 5 was identical; and Question 9(a) very similar. Anybody who studied last year's exam should have aced these questions, which comprise 59% of the exam.
2. Question 9(b) was actually very easy, but only one student got it!
3. In Question 10 the given four vectors form an orthogonal basis so you can use the Expansion Theorem.
4. Some students complained about the algebra in Question 13. Using row-reduction on an augmented system is actually the easiest way to solve the problem, and many students successfully did it.

Breakdown of Results: 47 students wrote this exam. The marks ranged from 30% to 89%, and the average was 67.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0%
A	25.5%	80-89%	25.5%
B	21.3%	70-79%	21.3%
C	25.5%	60-69%	25.5%
D	19.1%	50-59%	19.1%
F	8.6%	40-49%	4.3%
		30-39%	4.3%
		20-29%	0.0%
		10-19%	0.0%
		0-9%	0.0%



1. If U is a subspace of $\mathbb{R}^7$ and $\dim(U) = 4$, then $\dim\left(U^{\perp}\right) =$

   (a) 2

   (b) 3

   (c) 4

   (d) 5

Solution:

$$\dim\left(U^{\perp}\right) = 7 - \dim\left(U\right) = 7 - 4 = 3$$

The answer is (b).

2. In the solution to the system of equations $\left\{\begin{array}{rcrcrcr} 3x & + & 4y & - & 3z & = & 6 \\ x & + & 3y & + & z & = & -2 \\ -x & - & 2y & + & 4z & = & -8 \end{array}\right.$ the value of y is

   (a) 0

   (b) $\dfrac{6}{19}$

   (c) $-\dfrac{6}{19}$

   (d) 2

Solution: use Cramer's Rule.

$$y = \frac{\det\begin{bmatrix} 3 & 6 & -3 \\ 1 & -2 & 1 \\ -1 & -8 & 4 \end{bmatrix}}{\det\begin{bmatrix} 3 & 4 & -3 \\ 1 & 3 & 1 \\ -1 & -2 & 4 \end{bmatrix}} = \frac{0}{19} = 0$$

The answer is (a).

3. $\dim\left(\operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ -4 \\ 0 \end{bmatrix}\right\}\right) =$

   (a) 1

   (b) 2

   (c) 3

   (d) 4

Solution:

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -2 \\ -3 & 2 & 1 & -4 \\ 4 & -1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The answer is (c).

4. If T_1 is a reflection in the x-axis and T_2 is a reflection in the line $y = x$, then

(a) $T_1 \circ T_2$ is a reflection in the line $y = -x$.

(b) $T_1 \circ T_2$ is a reflection in the y-axis.

(c) $T_1 \circ T_2$ is a rotation through $\frac{\pi}{2}$, anti-clockwise, around the origin.

(d) $T_1 \circ T_2$ is a rotation through $\frac{\pi}{2}$, clockwise, around the origin.

Solution:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

which is a rotation matrix with

$$\theta = -\frac{\pi}{2}.$$

The answer is (d).

5. Suppose two lines in space are both parallel to the vector $\vec{d}$, one line passes through the point X_1, and the other line passes through the point X_2. The minimum distance between these two parallel lines is given by

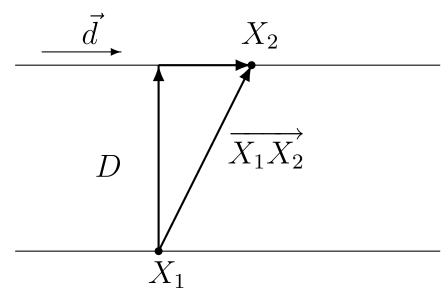
(a) $\left\| \overrightarrow{X_1X_2} \right\|$

(b) $\left\| \text{proj}_{\vec{d}} \left(\overrightarrow{X_1X_2} \right) \right\|$

(c) $\sqrt{\left\| \overrightarrow{X_1X_2} \right\|^2 - \left\| \text{proj}_{\vec{d}} \left(\overrightarrow{X_1X_2} \right) \right\|^2}$

(d) $\sqrt{\left\| \text{proj}_{\vec{d}} \left(\overrightarrow{X_1X_2} \right) \right\|^2 - \left\| \overrightarrow{X_1X_2} \right\|^2}$

Solution:



$$D^2 + \left\| \text{proj}_{\vec{d}} \left(\overrightarrow{X_1X_2} \right) \right\|^2 = \left\| \overrightarrow{X_1X_2} \right\|^2$$

The answer is (c).

6. Suppose $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 5y \\ 4x + 6y \end{bmatrix}$. Then the area of the image of the unit square is

(a) -8

(b) 8

(c) -32

(d) 32

Solution:

$$\left| \det \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix} \right| = |12 - 20| = 8.$$

The answer is (b).

7. Decide if the following statements are equivalent or not equivalent to the statement

$$A \text{ is an invertible } n \times n \text{ matrix.}$$

Give a brief, concise justification for your choice. Circle your choice.

(a) A is diagonalizable. **Equivalent** **Not equivalent**

Not equivalent: $A = 0$ is diagonalizable but not invertible.

(b) $A^T A$ is invertible. **Equivalent** **Not equivalent**

Equivalent: $\det(A^T A) = \det A^T \det A = (\det A)^2 = 0 \Leftrightarrow \det(A) = 0$.

(c) $\lambda = 0$ is an eigenvalue of A. **Equivalent** **Not equivalent**

Not equivalent: $A = 0$ has eigenvalue $\lambda = 0$ but is not invertible.

(d) $\mathrm{im}(A) = \mathbb{R}^n$. **Equivalent** **Not equivalent**

Equivalent: $\mathrm{im}(A) = \mathbb{R}^n$ if and only if the n columns of A are a basis of $\mathbb{R}^n$ if and only if A is invertible.

(e) $(\mathrm{row}(A))^\perp = \mathbb{R}^n$. **Equivalent** **Not equivalent**

Not equivalent: $(\mathrm{row}(0))^\perp = \{0\}^\perp = \mathbb{R}^n$, but $A = 0$ is not invertible.

8. Given that the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -3 \\ 0 & 0 & 1 & 3 & 11 \\ 1 & 2 & -3 & 4 & 0 \\ 1 & 2 & 4 & 2 & 8 \end{bmatrix} \text{ is } R = \begin{bmatrix} 1 & 2 & 0 & 0 & -6 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

state the rank of A, and find a basis for each of the following: the row space of A, the column space of A, and the null space of A.

Solution: the rank of A is 3, the number of leading 1's in R.

subspace	description of basis	vectors in basis
row$A$	three non-zero rows of R	$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \\ -6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\}$
	any three independent rows of A, but you must show they are independent	$\{R_1, R_2, R_3\}$ $\{R_1, R_2, R_4\}$ $\{R_2, R_3, R_4\}$ $\{R_1, R_3, R_4\}$
col A	any three independent columns of A	$\{C_1, C_3, C_4\}$ NB: $C_2 = 2C_1; C_5 = -6C_1 + 2C_3 + 3C_4$
null A	two basic solutions to $AX = 0$	$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ -3 \\ 1 \end{bmatrix} \right\}$

9. Let A be an $n \times n$ matrix. Let $U = \{X \in \mathbb{R}^n \mid A^T X = AX\}$.

(a) [5 marks] Show that U is a subspace of $\mathbb{R}^n$.

Solution: $U = \text{null}(A^T - A)$, so it is a subspace of $\mathbb{R}^n$.

(b) [5 marks] Show that if A is an orthogonal matrix, then $\dim U = \dim\, E_1(A^2)$.

Solution: Let $X \in U$:

$$\begin{aligned} A^{-1} = A^T \quad &\Rightarrow \quad A^T X = AX \Leftrightarrow A^{-1}X = AX \\ &\Rightarrow \quad X = A^2 X \Leftrightarrow X \in E_1(A^2) \end{aligned}$$

Thus $U = E_1(A^2)$, from which it follows that

$$\dim U = \dim\, E_1(A^2).$$

10. Write $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ as a linear combination of the four vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix}$.

Solution: Since

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} = 0, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} = 0, \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix} = 0$$

and

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} = 0, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix} = 0, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix} = 0$$

the given four vectors are an orthogonal basis of $\mathbb{R}^4$. Use the Expansion Theorem:

$$\begin{aligned}
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} &= \left(\frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + \left(\frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}} \right) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} \\
&\quad + \left(\frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}}{\begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix}} \right) \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} + \left(\frac{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix}}{\begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix}} \right) \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix} \\
&= 0 \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix} + \frac{27}{39} \begin{bmatrix} 3 \\ -1 \\ 2 \\ 5 \end{bmatrix} + \frac{21}{39} \begin{bmatrix} -2 \\ 5 \\ 3 \\ 1 \end{bmatrix}
\end{aligned}$$

11. Find an orthogonal matrix P and a diagonal matrix D such that $D = P^T AP$, if

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Solution:

Step 1: Find the eigenvalues of A.

$$\begin{aligned} \det(\lambda I - A) = \det \begin{bmatrix} \lambda-2 & 1 & 1 \\ 1 & \lambda-2 & 1 \\ 1 & 1 & \lambda-2 \end{bmatrix} &= \det \begin{bmatrix} \lambda-2 & 1 & 1 \\ 3-\lambda & \lambda-3 & 0 \\ 1 & 1 & \lambda-2 \end{bmatrix} \\ &= (\lambda-3)\det \begin{bmatrix} \lambda-2 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & \lambda-2 \end{bmatrix} \\ &= (\lambda-3)\det \begin{bmatrix} \lambda-1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & \lambda-2 \end{bmatrix} \\ &= (\lambda-3)(\lambda^2-3\lambda) \\ &= \lambda\,(\lambda-3)^2 \end{aligned}$$

So the eigenvalues of A are $\lambda_1 = 0$ and $\lambda_2 = 3$, repeated.

Step 2: Find mutually orthogonal eigenvectors of A.

$$E_3(A) = \text{null} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

$$E_0(A) = \text{null} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} = \text{null} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

OR: $E_0(A) = (E_3(A))^\perp$, which is the line passing through the origin, normal to the plane with equation $x + y + z = 0$.

Step 3: Divide each eigenvector by its length to get an orthonormal basis of eigenvectors, which are put into the columns of P. So

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \text{ and } D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

12. Let $U = \operatorname{span}\left\{X_1 = [\ -2 \quad 1 \quad 0 \quad 0\ ]^T, X_2 = [\ 1 \quad 1 \quad 1 \quad 2\ ]^T, X_3 = [\ 0 \quad 1 \quad 0 \quad 1\ ]^T\right\}$; let $X = \left[\ 1 \quad 1 \quad 2 \quad 1\ \right]^T$. Find:

(a) [6 marks] an orthogonal basis of U.

Solution: Use the Gram-Schmidt algorithm.

$$F_1 = X_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}; F_2 = X_2 - \frac{X_2 \cdot F_1}{\|F_1\|^2} F_1 = X_2 + \frac{1}{5} F_1 = \frac{1}{5} \begin{bmatrix} 3 \\ 6 \\ 5 \\ 10 \end{bmatrix};$$

$$F_3 = X_3 - \frac{X_3 \cdot F_1}{\|F_1\|^2} F_1 - \frac{X_3 \cdot F_2}{\|F_2\|^2} F_2 = X_3 - \frac{1}{5} F_1 - \frac{8}{85} F_2 = \frac{1}{17} \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}.$$

Optional: clear fractions and take

$$F_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, F_2 = \begin{bmatrix} 3 \\ 6 \\ 5 \\ 10 \end{bmatrix}, F_3 = \begin{bmatrix} 2 \\ 4 \\ -8 \\ 1 \end{bmatrix}.$$

Either way $\{F_1, F_2, F_3\}$ is an orthogonal basis of U.

(b) [6 marks] $\operatorname{proj}_U(X)$.

Solution: using $\{F_1, F_2, F_3\}$ with fractions cleared.

$$\begin{aligned} \operatorname{proj}_U X &= \frac{X \cdot F_1}{\|F_1\|^2} F_1 + \frac{X \cdot F_2}{\|F_2\|^2} F_2 + \frac{X \cdot F_3}{\|F_3\|^2} F_3 \\ &= -\frac{1}{5} F_1 + \frac{29}{170} F_2 - \frac{9}{85} F_3 \\ &= \frac{1}{10} \begin{bmatrix} 7 \\ 4 \\ 17 \\ 16 \end{bmatrix}. \end{aligned}$$

Cross-check/Alternate Solution: $U^\perp = \operatorname{span}\{Y\}$ with $Y = [\ 1 \quad 2 \quad 1 \quad -2\ ]^T$. Then

$$\operatorname{proj}_U X = X - \operatorname{proj}_{U^\perp}(X) = X - \frac{X \cdot Y}{\|Y\|^2} Y = X - \frac{3}{10} Y = \frac{1}{10} \begin{bmatrix} 7 \\ 4 \\ 17 \\ 16 \end{bmatrix}.$$

13. Find the least squares approximating quadratic for the data points

$$(2,0),(3,-10),(5,-48),(6,-76).$$

Solution: Let the least squares approximating quadratic be $y = a + bx + cx^2$; put

$$M = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 5 & 25 \\ 1 & 6 & 36 \end{bmatrix}, Z = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, Y = \begin{bmatrix} 0 \\ -10 \\ -48 \\ -76 \end{bmatrix}.$$

The normal equations are:

$$M^T M Z = M^T Y \Leftrightarrow \begin{bmatrix} 4 & 16 & 74 \\ 16 & 74 & 376 \\ 74 & 376 & 2018 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -134 \\ -726 \\ -4026 \end{bmatrix}$$

Approach 1: reduce the augmented matrix. This is not too bad:

$$\left[\begin{array}{ccc|c} 4 & 16 & 74 & -134 \\ 16 & 74 & 376 & -726 \\ 74 & 376 & 2018 & -4028 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 8 & 37 & -67 \\ 0 & 10 & 80 & -190 \\ 0 & 80 & 649 & -1547 \end{array}\right] \to \left[\begin{array}{ccc|c} 2 & 0 & -27 & 85 \\ 0 & 1 & 8 & -19 \\ 0 & 0 & 9 & -27 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right].$$

Approach 2: use the adjoint method, and your calculator, to find the inverse of $M^T M$. The numbers get quite big:

$$(M^T M)^{-1} = \frac{1}{360} \begin{bmatrix} 7956 & -4464 & 540 \\ -4464 & 2596 & -320 \\ 540 & -320 & 40 \end{bmatrix};$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{360} \begin{bmatrix} 7956 & -4464 & 540 \\ -4464 & 2596 & -320 \\ 540 & -320 & 40 \end{bmatrix} \begin{bmatrix} -134 \\ -726 \\ -4026 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$$

Answer: the least squares approximating quadratic is

$$y = 2 + 5x - 3x^2.$$

Aside: this quadratic polynomial actually passes through each of the given data points. That is, the system of equations $MZ = Y$ is consistent, and the normal equations are not required. But, who knew?