University of Toronto
MAT187H1F Quiz 4
Thursday, June 20, 2013
Solutions, with guide to part marks. No half marks, please.
Duration: 30 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

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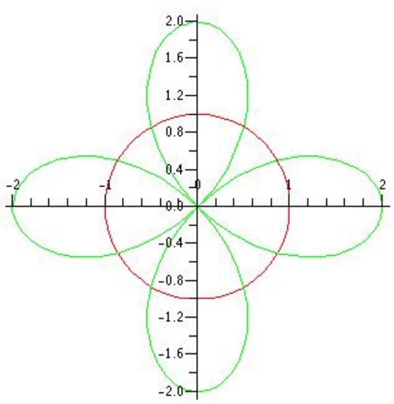
1. [10 marks] Find the area of the region inside the four leaved rose with polar equation $r = 2\cos(2\theta)$ but outside the circle with polar equation $r = 1$.

Solution:



Solve $1 = 2\cos 2\theta$ for intersection points:

$$2\cos(2\theta) = 1 \;\Rightarrow\; \cos(2\theta) = \frac{1}{2}$$
$$\Rightarrow\; 2\theta = \pm\frac{\pi}{3}, \pm\frac{7\pi}{3}$$
$$\Rightarrow\; \theta = \pm\frac{\pi}{6}, \pm\frac{7\pi}{6}$$

Note: by symmetry not all intersection points are needed. But should one want them, one needs to also solve

$$2\cos(2\theta) = -1 \Leftrightarrow \theta = \pm\frac{\pi}{3}, \pm\frac{4\pi}{3}$$

By symmetry,

$$\begin{aligned}
A &= 4\left(\frac{1}{2}\int_{-\pi/6}^{\pi/6} (r^2 - 1)\, d\theta\right) \\
&= 8\left(\frac{1}{2}\int_{0}^{\pi/6} (r^2 - 1)\, d\theta\right) \\
&= 4\int_0^{\pi/6} (4\cos^2 2\theta - 1)\, d\theta \\
&= 4\int_0^{\pi/6} (2 + 2\cos(4\theta) - 1)\, d\theta \\
&= 4\int_0^{\pi/6} (1 + 2\cos(4\theta))\, d\theta \\
&= [4\theta + 2\sin(4\theta)]_0^{\pi/6} \\
&= \frac{2\pi}{3} + \sqrt{3}
\end{aligned}$$

2. [10 marks] Find the arc length of the curve

$$\mathbf{r}(t) = t^2\,\mathbf{i} + (\cos t + t\,\sin t)\,\mathbf{j} + (\sin t - t\,\cos t)\,\mathbf{k}$$

for $0 \le t \le \pi$.

Solution:

$$\begin{aligned}
\frac{d\mathbf{r}}{dt} &= 2t\,\mathbf{i} + (-\sin t + \sin t + t\,\cos t)\,\mathbf{j} + (\cos t - \cos t + t\,\sin t)\,\mathbf{k} \\
&= 2t\,\mathbf{i} + t\,\cos t\,\mathbf{j} + t\,\sin t\,\mathbf{k} \\
\Rightarrow \left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{4t^2 + t^2\cos^2 t + t^2\sin^2 t} \\
&= \sqrt{5}\,t
\end{aligned}$$

Thus the length of the curve is

$$L = \int_0^\pi \left\|\frac{d\mathbf{r}}{dt}\right\| dt = \int_0^\pi \sqrt{5}\,t\,dt = \left[\frac{\sqrt{5}\,t^2}{2}\right]_0^\pi = \frac{\sqrt{5}\,\pi^2}{2}$$