MAT187H1F - Calculus II - Spring 2017

Solutions for Term Test 1 - May 31, 2017

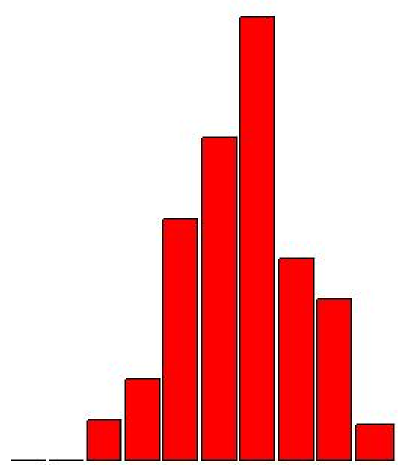
Time allotted: 110 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Comments:

1. Every part of the test was done perfectly at least once. However, there was no single perfect paper. The only question with a failing average was Question 8.
2. The average was 'only' 61.5%. For the purposes of your term mark calculations, I will count this test out of 76 (which was the top mark), effectively making the average on this test 64.7%

Breakdown of Results: 39 students wrote this test, including one student not officially registered in the course. The marks ranged from 26% to 95%, and the average was 61.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.1%
A	15.4%	80-89%	10.3%
B	12.8%	70-79%	12.8%
C	28.2%	60-69%	28.2%
D	20.5%	50-59%	20.5%
F	23.1%	40-49%	15.4%
		30-39%	5.1%
		20-29%	2.6%
		10-19%	0.0%
		0-9%	0.0%



PART I : No explanation is necessary. Answer all parts of Questions 1 and 2 by putting your answer in the appropriate blank or by circling your choice(s).

1: [avg: 5.05/10]

1.(a) [2 marks] The exact value of $\displaystyle\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx$ is: $\underline{\qquad \pi/2 \qquad}$

$$\int_0^2 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2 = \sin^{-1}\left(\frac{2}{2}\right) = \frac{\pi}{2}.$$

1.(b) [6 marks] Consider the integral $\displaystyle\int_0^{\pi/3} \sin^{37} x \cos^{10} x\,dx$. Use a substitution to obtain

$$\int_0^{\pi/3} \sin^{37} x \cos^{10} x\,dx = \int_a^b f(u)\,du,$$

where $f(u)$ is a polynomial function of u, and $a < b$. Then

$u = \underline{\quad \cos x \quad}$

$a = \underline{\quad 1/2 \quad}$

$b = \underline{\quad 1 \quad}$

$f(u) = \underline{\quad u^{10}(1-u^2)^{18} \quad}$

(No need to expand!)

1.(c) [2 marks] A population of bacteria growing exponentially has a doubling time of 15 minutes. How long will it take for the population of bacteria to become eight times its original size?

Quick way: since $8 = 2^3$, it will take three doubling periods: $3 \times 15 = 45$ minutes.

Answer: $\underline{\qquad 45 \text{ min} \qquad}$

2: [avg: 8.31/10]

2.(a) [6 marks] Consider the rational function

$$\frac{x^4 - 10x^2 + x - 1}{(x^3 + x^2)(x^2 - x + 1)(x^2 + 9)^2}.$$

To find its partial fraction decompostion we write this function as a sum of the following terms (circle all that apply):

$\boxed{\text{(a)}}$	$\frac{A}{x}$	$\boxed{\text{(b)}}$	$\frac{D}{x+1}$	$\boxed{\text{(c)}}$	$\frac{G+Hx}{x^2+9}$	$\boxed{\text{(d)}}$	$\frac{Mx+N}{x^2-x+1}$	(e)	$\frac{Sx+T}{x^3+x^2}$
$\boxed{\text{(f)}}$	$\frac{B}{x^2}$	(g)	$\frac{E}{(x+1)^2}$	$\boxed{\text{(h)}}$	$\frac{I+Jx}{(x^2+9)^2}$	(i)	$\frac{Ox+P}{(x^2-x+1)^2}$	(j)	$\frac{Ux+V}{(x^3+x^2)^2}$
(k)	$\frac{C}{x^3}$	(l)	$\frac{F}{(x+1)^3}$	(m)	$\frac{K+Lx}{(x^2+9)^3}$	(n)	$\frac{Qx+R}{(x^2-x+1)^3}$	(o)	$\frac{Wx+X}{(x^3+x^2)^3}$

2.(b) [2 marks] Circle all of the following differential equations which are **not** separable.

$$\boxed{\text{(a)}} \quad \frac{dy}{dx} = \ln(x^2\,y^2) \quad (b) \quad \frac{dy}{dx} = e^{x+y} \quad (c) \quad \frac{dy}{dx} = \sqrt{x^4 + x^4y^2}$$

2.(c) [2 marks] Consider the initial value problem

$$\text{DE: } \frac{dP}{dt} = \frac{P}{5} - \frac{P^2}{75}, \quad \text{IC: } P(0) = 5.$$

The value of $\lim_{t\to\infty} P$ is: 15

$$\frac{dP}{dt} = \frac{P}{5} - \frac{P^2}{75} = \frac{P}{5}\left(1 - \frac{P}{15}\right)$$

PART II : Present **complete** solutions to the following questions in the space provided.

3. [10 marks; avg: 9.0/10] Newton's Law of Cooling states that

$$\frac{dT}{dt} = -k(T - A),$$

where T is the temperature of an object at time t, in a room with constant ambient temperature A, and k is a positive constant.

Suppose a cup of coffee, initially at temperature 95 C, is placed on a table in a room with constant temperature 22 C. If the temperature of the coffee is 70 C five minutes later, when will the temperature of the coffee be 30 C? Sketch a graph of T for $t \geq 0$, labeling any horizontal asymptotes.

Solution I: we have $A = 22$ and $T_0 = 95$, so if you memorized the solution you know that

$$T = 22 + 73e^{-kt}.$$

To find k, let $t = 5$ and $T = 70$:

$$70 = 22 + 73e^{-5k} \Leftrightarrow \frac{48}{73} = e^{-5k} \Leftrightarrow e^{5k} = \frac{73}{48} \Leftrightarrow k = \frac{1}{5}\ln\left(\frac{73}{48}\right) \approx 0.083851686$$

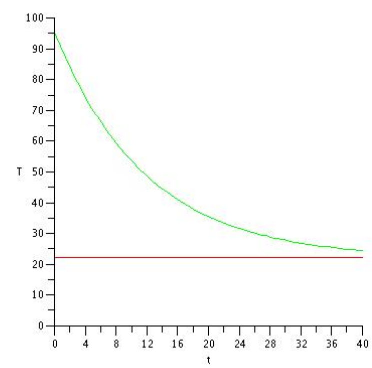
Now let $T = 30$ and solve for t:

$$30 = 22 + 73e^{-kt} \Leftrightarrow \frac{8}{73} = e^{-kt}$$

$$\Leftrightarrow e^{kt} = 9.125 \Leftrightarrow kt = \ln 9.125 \Leftrightarrow t = \frac{5\ \ln 9.125}{\ln(73/48)} \approx 26.4$$

So it will take about 26.4 min for the coffee to reach 30 C.

The graph is to the right. $T = 22$ is a horizontal asymptote.



Solution II: separate variables. NB: it is not necessary to solve for T.

$$\int \frac{dT}{T - 22} = -\int k\,dt \Leftrightarrow \ln(T - 22) = -kt + C.$$

To find C, let $t = 0, T = 95$: $C = \ln 73$. Now to find k, let $t = 5, T = 70$:

$$\ln 48 = -5k + \ln 73 \Leftrightarrow 5k = \ln 73 - \ln 48 \Leftrightarrow k = \frac{1}{5}\ln(73/48).$$

Finally, let $T = 30$ and solve for t:

$$\ln 8 = -\frac{t\ \ln(73/48)}{5} + \ln 73 \Leftrightarrow \frac{t\ln(73/48)}{5} = \ln 73 - \ln 8 \Leftrightarrow t = \frac{5\ln(73/8)}{\ln(73/48)},$$

as before.

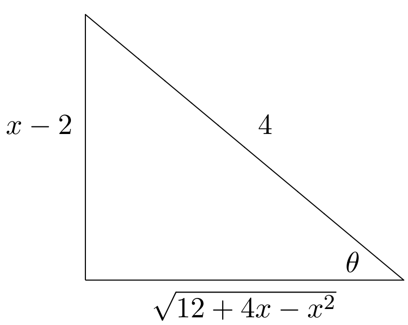
4. [10 marks: avg: 5.97/10] Find $\displaystyle\int \frac{(4+3x)}{\sqrt{12+4x-x^2}}\,dx$.

Solution: complete the square and use a trig substitution.

$$12+4x-x^2 = -(x^2-4x)+12 = -(x^2-4x+4-4)+12 = 16-(x-2)^2.$$

Let $x-2 = 4\sin\theta$. Then $x = 2+4\sin\theta$ and $dx = 4\cos\theta\,d\theta$. So

$$\begin{aligned}
\int \frac{(4+3x)}{\sqrt{12+4x-x^2}}\,dx &= \int \frac{(4+6+12\sin\theta)}{\sqrt{16-16\sin^2\theta}}\,4\cos\theta\,d\theta \\
&= \int \frac{(10+12\sin\theta)}{4\cos\theta}\,4\cos\theta\,d\theta \\
&= \int (10+12\sin\theta)\,d\theta \\
&= 10\,\theta - 12\cos\theta + C \\
&= 10\sin^{-1}\left(\frac{x-2}{4}\right) - 12\left(\frac{\sqrt{12+4x-x^2}}{4}\right) + C \\
&= 10\sin^{-1}\left(\frac{x-2}{4}\right) - 3\sqrt{12+4x-x^2} + C
\end{aligned}$$



where we used the triangle to the left with

$$\sin\theta = \frac{x-2}{4}$$

to get

$$\cos\theta = \frac{\sqrt{12+4x-x^2}}{4}.$$

5. [avg: 7.74/10] Find the general solution, y in terms of x, for each of the following differential equations:

(a) [5 marks] $\dfrac{dy}{dx} = e^{-y}\cos^2 x$

Solution: separate variables.

$$\int e^y\,dy = \int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos(2x))\,dx$$

$$\Rightarrow e^y = \frac{x}{2} + \frac{\sin(2x)}{4} + C$$

$$\Rightarrow y = \ln\left(\frac{x}{2} + \frac{\sin(2x)}{4} + C\right)$$

(b) [5 marks] $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{1}{x^3}$, for $x > 0$.

Solution: use the method of the integrating factor.

$$\mu = e^{\int \frac{1}{x}\,dx} = e^{\ln(x)} = x.$$

Then

$$y = \frac{1}{\mu}\int \mu\left(\frac{1}{x^3}\right)dx = \left(\frac{1}{x}\right)\int\left(\frac{1}{x^2}\right)dx = \left(\frac{1}{x}\right)\left(-\frac{1}{x} + C\right) = -\frac{1}{x^2} + \frac{C}{x}$$

6. [10 marks; avg: 5.1/10] Find the exact value of $\displaystyle\int_1^\infty \frac{\tan^{-1} x}{x^2}\, dx$.

Solution: use integration by parts and the definition of improper integral. Let

$$u = \tan^{-1} x;\ dv = \frac{dx}{x^2}.$$

Then

$$du = \frac{1}{1+x^2}\, dx; \quad v = -\frac{1}{x}$$

and

$$\begin{aligned}
\int_1^\infty \frac{\tan^{-1} x}{x^2}\, dx &= \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{x^2}\, dx \\
&= \lim_{b\to\infty} [uv]_1^b - \lim_{b\to\infty} \int_1^b v\, du \\
&= \lim_{b\to\infty} \left[-\frac{\tan^{-1} x}{x}\right]_1^b - \lim_{b\to\infty} \int_1^b \frac{1}{(-x)(1+x^2)}\, dx \\
&= -\lim_{b\to\infty} \left(\frac{\tan^{-1} b}{b}\right) + \tan^{-1} 1 + \lim_{b\to\infty} \int_1^b \frac{1}{x(1+x^2)}\, dx \\
(\text{let } x = \tan\theta) \quad &= 0 + \frac{\pi}{4} + \int_{\pi/4}^{\pi/2} \frac{1}{\tan\theta(1+\tan^2\theta)} \sec^2\theta\, d\theta \\
&= \frac{\pi}{4} + \int_{\pi/4}^{\pi/2} \cot\theta\, d\theta \\
&= \frac{\pi}{4} + [\ln(\sin\theta)]_{\pi/4}^{\pi/2} \\
&= \frac{\pi}{4} + \ln 1 - \ln(1/\sqrt{2}) \\
&= \frac{\pi}{4} + \frac{\ln 2}{2}
\end{aligned}$$

OR use partial fractions to evaulate $\displaystyle\int_1^b \frac{1}{x(1+x^2)}\, dx$:

$$\begin{aligned}
\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2} \quad &\Rightarrow \quad 1 = A + Ax^2 + Bx^2 + Cx \\
&\Rightarrow \quad A = 1,\ B = -1,\ C = 0 \\
&\Rightarrow \quad \int_1^b \frac{1}{x(1+x^2)}\, dx = \int_1^b \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx \\
&\Rightarrow \quad \int_1^b \frac{1}{x(1+x^2)}\, dx = \left[\ln x - \frac{1}{2}\ln(1+x^2)\right]_1^b \\
&\Rightarrow \quad \int_1^b \frac{1}{x(1+x^2)}\, dx = \ln b - \ln\sqrt{1+b^2} + \frac{\ln 2}{2} \\
&\Rightarrow \quad \lim_{b\to\infty} \int_1^b \frac{1}{x(1+x^2)}\, dx = \lim_{b\to\infty} \ln\left(\frac{b}{\sqrt{1+b^2}}\right) + \frac{\ln 2}{2} = \ln 1 + \frac{\ln 2}{2} = \frac{\ln 2}{2}
\end{aligned}$$

7. [10 marks; avg: 5.87/10] Recall: if $x(t)$ is the mass of solute dissolved in a solution of volume $V(t)$, at time t, in a large mixing tank, then

$$\frac{d\,x(t)}{dt} + \frac{r_o\,x(t)}{V(t)} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 250 L of pure water. At $t = 0$, a salt water solution containing 0.25 kg salt per L is added at a rate of 20 L/min and the mixed solution is drained off at a rate of 10 L/min. How much salt is in the tank when it reaches the point of overflowing?

Solution: We have $r_i = 20, r_0 = 10, c_i = 0.25 = 1/4$, so $V = 250 + (r_i - r_0)t = 250 + 10t$. Hence the differential equation is

$$\frac{dx}{dt} + \frac{10\,x}{250 + 10\,t} = \frac{20}{4} \Leftrightarrow \frac{dx}{dt} + \frac{x}{25+t} = 5.$$

Use the method of the integrating factor, with

$$\mu = e^{\int \frac{dt}{25+t}} = e^{\ln(25+t)} = 25 + t.$$

Then

$$x = \frac{1}{(25+t)} \int 5(25+t)\,dt = \frac{1}{(25+t)} \left(\frac{5\,(25+t)^2}{2} + C \right) = \frac{5\,(25+t)}{2} + \frac{C}{(25+t)}.$$

At $t = 0, x = 0$, so

$$0 = \frac{5(25)}{2} + \frac{C}{25} \Rightarrow C = -\frac{5^5}{2}.$$

Finally, $V = 1000 \Leftrightarrow 250 + 10t = 1000 \Leftrightarrow t = 75$, and at that time:

$$x = \frac{5(100)}{2} - \frac{5^3}{2(100)} = 234.375,$$

exactly. So there will be 234.375 kg of salt in the tank when it reaches the point of overflowing.

8. [10 marks; avg: 2.11/10] The Gompertz growth model with $P > 0$ has differential equation

$$\frac{dP}{dt} = -r\,P\,\ln\left(\frac{P}{K}\right),$$

for positive constants r and K.

(a) [4 marks] Find the equilibrium solution to the Gompertz model and determine if it is stable or unstable.

Solution: since r and P are both positive,

$$\frac{dP}{dt} = 0 \Rightarrow \ln\left(\frac{P}{K}\right) = 0 \Rightarrow P = K.$$

Thus the equilibrium solution is $P = K$. It is a stable equilibrium since

1. if $P > K$, then $\ln\left(\frac{P}{K}\right) > \ln 1 > 0$, so

$$\frac{dP}{dt} = -r\,P\,\ln\left(\frac{P}{K}\right) < 0$$

and the value of P will decrease, back towards the equilibrium value K.

2. On the other hand, if $P < K$, then $\ln\left(\frac{P}{K}\right) < \ln 1 < 0$, so

$$\frac{dP}{dt} = -r\,P\,\ln\left(\frac{P}{K}\right) > 0$$

and the value of P will increase, back towards the equilibrium value K.

(b) [6 marks] Solve the Gompertz differential equation if $r = 1, K = 100$, and $P = 20$ when $t = 0$.

Solution: we have $r = 1$ and $K = 100$, so

$$\frac{dP}{dt} = -P \ln\left(\frac{P}{100}\right) = -P\,(\ln P - \ln 100).$$

Separate variables:

$$\begin{aligned}
\frac{dP}{dt} = -P\,(\ln P - \ln 100) \quad &\Rightarrow \quad \int \frac{dP}{P(\ln P - \ln 100)} = -\int dt \\
(\text{let } u = \ln P - \ln 100) \quad &\Rightarrow \quad \int \frac{du}{u} = -t + C \\
&\Rightarrow \quad \ln|u| = -t + C \\
&\Rightarrow \quad \ln|\ln P - \ln 100| = -t + C
\end{aligned}$$

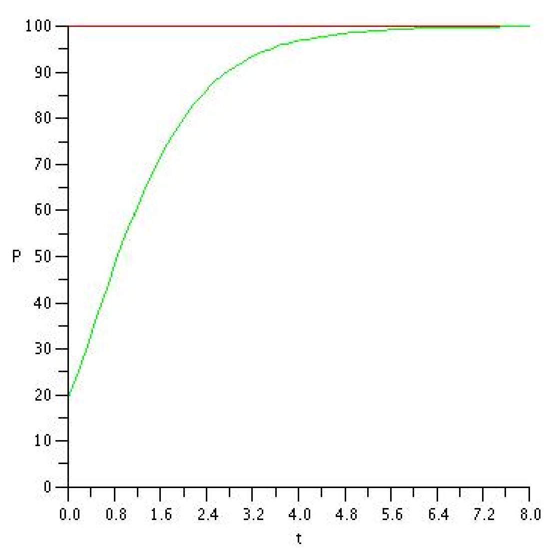
To find C use the initial condition:

$$\ln|\ln 20 - \ln 100| = 0 + C \Leftrightarrow C = \ln|\ln 20 - \ln 100| = \ln|-\ln 5| = \ln(\ln 5)$$

Now solve for P :

$$\begin{aligned}
\ln|\ln P - \ln 100| = -t + \ln(\ln 5) \quad &\Rightarrow \quad \left|\ln\left(\frac{P}{100}\right)\right| = e^{-t+\ln(\ln 5)} = e^{-t}\,\ln 5 \\
(\text{since } P_0 = 20 < 100) \quad &\Rightarrow \quad \ln\left(\frac{P}{100}\right) = -e^{-t}\,\ln 5 \\
&\Rightarrow \quad \frac{P}{100} = e^{-e^{-t}\,\ln 5} = 5^{-e^{-t}} \\
&\Rightarrow \quad P = 100 \cdot 5^{-e^{-t}} \text{ or } P = \frac{100}{5^{e^{-t}}}
\end{aligned}$$

For interest, the graph of P is here:



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