MAT187H1F - Calculus II - Spring 2015

Solutions to Term Test 1 - May 27, 2015

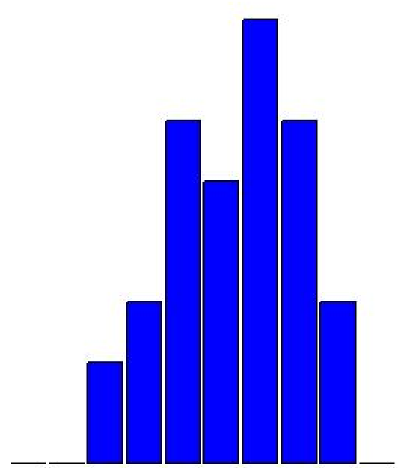
Time allotted: 100 minutes. Aids permitted: Casio FX-991 or Sharp EL-520 calculator.

General Comments:

1. A third of the class failed this test. In particular, Questions 11, 13, 15 and 16 had failing averages. But the integrals in Question 11 and 13(b) were completely *routine,* in the sense that the methods we covered *work*. No *creative* steps are required!
2. In Question 15, part (a) was based on WeBWorK problems and was actually very straightforward. Finding the equilibrium solutions in part (b) is also straightforward. The only 'hard' question on the whole test was the second part of part (b), since it requires a little figuring out to determine if each equilibrium solution is stable or unstable.
3. Question 16 is a completely routine logistic growth problem and should have been aced!
4. Question 13, part (a) was basically a repeat of a question 10 on the mat187 test of Feb. 3, 2015. It is 'slick', but not hard.

Breakdown of Results: all 91 students wrote this test. The marks ranged from 22.5% to 86.25%, and the average was only 57.25%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right. Note the bimodal distribution.

Grade	%	Decade	%
		90-100%	0.0%
A	8.8%	80-89%	8.8%
B	18.7%	70-79%	18.7%
C	24.2%	60-69%	24.2%
D	15.4%	50-59%	15.4%
F	32.9%	40-49%	18.7%
		30-39%	8.8%
		20-29%	5.4%
		10-19%	0.0%
		0-9%	0.0%



PART I : No explanation is necessary.

Answer all of the following ten short-answer questions by putting your answer in the appropriate blank or by circling your choice(s). Average total on this page: 7.13/10

1. [1 mark] Let $f(x) = x^3 + x,\ g(x) = x^3 + x^2 - 3x$. Let A equal the total area of the region between the graphs of $y = f(x)$ and $y = g(x)$ on the interval $[-1, 4]$. Then $A =$

   (a) $\displaystyle\int_{-1}^{4} (f(x) - g(x))\, dx$ $\qquad$ (c) $\displaystyle\int_{-1}^{4} (g(x) - f(x))\, dx$

   (b) $\displaystyle\int_{-1}^{0} (f(x) - g(x))\, dx + \int_{0}^{4} (g(x) - f(x))\, dx$ $\qquad$ $\boxed{\text{(d)}}$ $\displaystyle\int_{-1}^{0} (g(x) - f(x))\, dx + \int_{0}^{4} (f(x) - g(x))\, dx$

2. [5 marks] Consider the integral $\displaystyle\int_0^{\pi/6} \sin^{42} x \cos^9 x\, dx$; use a substitution to obtain

$$\int_0^{\pi/6} \sin^{42} x \cos^9 x\, dx = \int_a^b f(u)\, du.$$

   Then

$$u = \underline{\quad \sin x \quad}$$

$$a = \underline{\quad 0 \quad}$$

$$b = \underline{\quad 1/2 \quad}$$

$$f(u) = \underline{\quad u^{42}(1 - u^2)^4 \quad}$$

3. [2 marks] A population of bacteria is growing exponentially so that its doubling time is 45 min. To the nearest minute, how long will it take for a population of this bacteria to triple in size? $\underline{\quad 71 \quad}$

4. [1 mark] Consider two functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge 1$.

   Assume that $\displaystyle\int_1^{\infty} g(x)\, dx$ converges. What can you say about $\displaystyle\int_1^{\infty} f(x)\, dx$?

   $\boxed{\text{(a)}}$ It converges. (b) It diverges. (c) There is not enough information to decide.

5. [1 mark] Consider two functions $f(x)$ and $g(x)$ satisfying $0 \le f(x) \le g(x)$ for $x \ge 1$.

   Assume that $\displaystyle\int_1^{\infty} f(x)\, dx$ converges. What can you say about $\displaystyle\int_1^{\infty} g(x)\, dx$?

   (a) It converges. (b) It diverges. $\boxed{\text{(c)}}$ There is not enough information to decide.

Average total on this page: 8.27/10

6. [6 marks] Consider the rational function

$$\frac{x^4 - 10x^2 + x - 1}{(x+2)^3(x-1)(x^2+4)^2}.$$

To find its partial fraction decompostion we write this function as a sum of the following terms (circle all that apply):

$\boxed{(a)}$ $\frac{A}{x+2}$ $\boxed{(b)}$ $\frac{D}{x-1}$ $\boxed{(c)}$ $\frac{G+Hx}{x^2+4}$ (d) $\frac{M}{x^2+4}$ (e) $\frac{P}{x-2}$

$\boxed{(f)}$ $\frac{B}{(x+2)^2}$ (g) $\frac{E}{(x-1)^2}$ $\boxed{(h)}$ $\frac{I+Jx}{(x^2+4)^2}$ (i) $\frac{N}{(x^2+4)^2}$ (j) $\frac{Q}{(x-2)^2}$

$\boxed{(k)}$ $\frac{C}{(x+2)^3}$ (l) $\frac{F}{(x-1)^3}$ (m) $\frac{K+Lx}{(x^2+4)^3}$ (n) $\frac{O}{(x^2+4)^3}$ (o) $\frac{R}{(x-2)^3}$

7. [1 mark] Which of the following differential equations is **not** separable?

$$(a)\ \frac{dy}{dx} = \frac{4y^2+1}{x^2+9} \quad \boxed{(b)}\ \frac{dy}{dx} = x^2 + y^3 \quad (c)\ \frac{dy}{dx} = 4x - 3yx$$

8. [1 mark] Consider the initial value problem

$$\text{DE: } \frac{dP}{dt} = 0.03\,P - 0.00012\,P^2, \quad \text{IC: } P(0) = 40.$$

The value of $\lim_{t\to\infty} P$ is: 250

9. [1 mark] Consider the differential equation $\frac{dy}{dx} = x^2 + y^2$ and its direction field (same as slope field.)

The curves in the direction field along which the slopes are constant are

(a) vertical lines. (b) horizontal lines $\boxed{(c)}$ concentric circles. (d) parabolas.

10. [1 mark] $\int_0^\infty e^{-5x}\,dx =$ 1/5

PART II : Present **complete** solutions to the following questions in the space provided.

11. [10 marks; avg: 4.86/10] Find the exact value of $\displaystyle\int_0^3 x \tan^{-1}\sqrt{x}\,dx$.

Solution: let $x = t^2$ and then use integration by parts. (Or use parts first, and then let $x = t^2$.)

$$\begin{aligned}
\int_0^3 x \tan^{-1}\sqrt{x}\,dx &= \int_0^{\sqrt{3}} t^2 \tan^{-1} t\,(2t\,dt) \\
&= \int_0^{\sqrt{3}} 2t^3 \tan^{-1} t\,dt \\
(\text{let } u = \tan^{-1} t, dv = 2t^3\,dt) \quad &= [uv]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} v\,du \\
&= \frac{1}{2}\left[t^4 \tan^{-1} t\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{t^4\,dt}{1+t^2} \\
(\text{by long division}) \quad &= \frac{1}{2}\left[t^4 \tan^{-1} t\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \left(t^2 - 1 + \frac{1}{1+t^2}\right) dt \\
&= \frac{1}{2}\left[t^4 \tan^{-1} t\right]_0^{\sqrt{3}} - \frac{1}{2}\left[\frac{t^3}{3} - t + \tan^{-1} t\right]_0^{\sqrt{3}} \\
&= \left(\frac{9}{2}\right)\left(\frac{\pi}{3}\right) - \frac{3\sqrt{3}}{6} + \frac{\sqrt{3}}{2} - \left(\frac{1}{2}\right)\left(\frac{\pi}{3}\right) \\
&= \frac{4\pi}{3}
\end{aligned}$$

12. [10 marks; avg: 6.2/10] Find the exact value of $\displaystyle\int_2^4 \frac{(11-x)\,dx}{(-5+6x-x^2)^{3/2}}$.

Solution: complete the square and use a trig substitution.

$$-5+6x-x^2 = -(x^2-6x+9-9)-5 = -(x^2-6x+9)+9-5 = 2^2-(x-3)^2.$$

Let $x-3 = 2\sin\theta$. Then $x = 3+2\sin\theta$ and $dx = 2\cos\theta\,d\theta$, so

$$\begin{aligned}
\int_2^4 \frac{(11-x)\,dx}{(-5+6x-x^2)^{3/2}} &= \int_{-\pi/6}^{\pi/6} \frac{(11-3-2\sin\theta)}{(4-4\sin^2\theta)^{3/2}}\,2\cos\theta\,d\theta \\
&= \frac{1}{4}\int_{-\pi/6}^{\pi/6} \frac{(8-2\sin\theta)}{\cos^3\theta}\,\cos\theta\,d\theta \\
&= \frac{1}{4}\int_{-\pi/6}^{\pi/6} \frac{(8-2\sin\theta)}{\cos^2\theta}\,d\theta \\
&= 2\int_{-\pi/6}^{\pi/6} \sec^2\theta\,d\theta - \frac{1}{2}\int_{-\pi/6}^{\pi/6} \tan\theta\,\sec\theta\,d\theta \\
&= 4\left[\tan\theta\right]_0^{\pi/6} - 0 \\
&= 4\left(\frac{1}{\sqrt{3}}\right) \\
&= \frac{4}{\sqrt{3}}
\end{aligned}$$

Note: I used

$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

if f is even, and

$$\int_{-a}^{a} f(x)\,dx = 0,$$

if f is odd. But the calculations aren't much worse if you don't do it this way.

13. [10 marks: avg: 3.93/10] The parts of this question are unrelated.

(a) [4 marks] Suppose the function $f(x)$ is defined on the interval $[-a,a]$ and is a solution to the initial value problem

$$\text{DE: } f''(x) = \sqrt{1+(f'(x))^2}, \text{ IC: } f'(a) = -f'(-a) = 5.$$

Find the length of the curve $y = f(x)$ on the interval $[-a,a]$.

Solution:

$$L = \int_{-a}^{a} \sqrt{1+(f'(x))^2}\,dx = \int_{-a}^{a} f''(x)\,dx = \big[f'(x)\big]_{-a}^{a} = f'(a) - f'(-a) = 5+5 = 10.$$

(b) [6 marks] Find the exact value of $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^2}$.

Solution: let $x = \tan\theta$ and transform the improper integral into a proper integral.

$$\begin{aligned}
\int_0^\infty \frac{dx}{(1+x^2)^2} &= \int_0^{\pi/2} \frac{\sec^2\theta}{(1+\tan^2\theta)^2}\,d\theta \\
&= \int_0^{\pi/2} \frac{\sec^2\theta}{\sec^4\theta}\,d\theta \\
&= \int_0^{\pi/2} \cos^2\theta\,d\theta \\
\text{(use double angle formula)} &= \int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\,d\theta \\
&= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\
&= \frac{\pi}{4} + 0 - 0 - 0 \\
&= \frac{\pi}{4}
\end{aligned}$$

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14. [10 marks; avg: 7.27/10] Newton's Law of Cooling states that

$$\frac{dT}{dt} = -k(T - A),$$

where T is the temperature of an object at time t, in a room with constant ambient temperature A, and k is a positive constant.

Suppose a cup of coffee, initially at temperature 85 C, is placed on a table in a room with constant temperature 20 C. If the temperature of the coffee is 70 C five minutes later, when will the temperature of the coffee be 50 C? Sketch a graph of T for $t \geq 0$, labeling any horizontal asymptotes.

Solution: use $A = 20$ and separate variables. (Note: it is not actually necessary to solve for T.)

$$\begin{aligned} \frac{dT}{dt} = -k(T-20) \quad &\Leftrightarrow \quad \int \frac{dT}{T-20} = -\int k\,dt \\ &\Leftrightarrow \quad \ln|T-20| = -k\,t + C \end{aligned}$$

To find C, let $t = 0$ and $T = 85$: $C = \ln 65$. To find k, let $t = 5$ and $T = 70$:

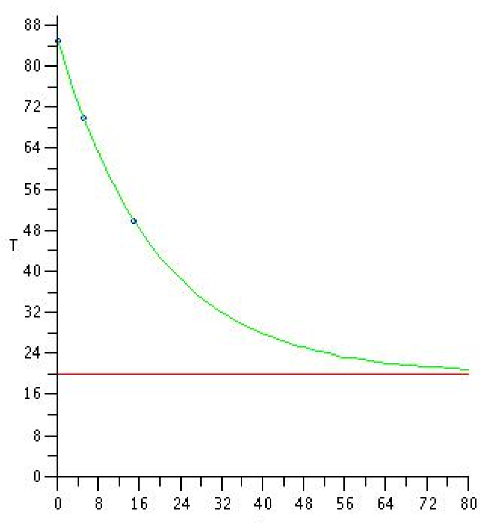
$$\ln 50 = -5\,k + \ln 65 \Leftrightarrow k = \frac{1}{5}\ln\left(\frac{13}{10}\right) \approx 0.05247$$

Now let $T = 50$ and solve for t:

$$\ln 30 = -\frac{t}{5}\ln\left(\frac{13}{10}\right) + \ln 65 \Leftrightarrow t = \frac{5\ln(13/6)}{\ln(13/10)} \approx 14.735$$

So the temperature of the coffee will have decreased from 85 C to 50 C in about 14.7 minutes.

Graph: your graph must pass through the three indicated points, $(0, 85), (5, 70)$ and $(14.7, 50)$, and be asymptotic to $T = 20$.



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15. [10 marks; avg: 4.12/10] The parts of this question are unrelated.

(a) [4 marks] Find all values of r such that $y = x^r$ is a solution to the differential equation

$$x^2\, y'' + 6x\, y' - 6\, y = 0.$$

Solution: substitute $y = x^r$, $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$ into given DE and solve for r:

$$x^2\left(r(r-1)x^{r-2}\right) + 6x\left(rx^{r-1}\right) - 6\, x^r = 0 \Leftrightarrow \left(r^2 + 5r - 6\right) x^r = 0.$$

Assume $y = x^r \neq 0$ to find non-zero solutions y. Then

$$r^2 + 5r - 6 = 0 \Leftrightarrow (r+6)(r-1) = 0 \Leftrightarrow r = -6 \text{ or } r = 1.$$

(b) [6 marks] Suppose that y is a function of t and $y' = y^3 - 5y^2 + 6y$ for $t \geq 0$. Find all the equilibrium solutions to this differential equation and determine if each equilibrium solution is stable or unstable.

Solution: set $y' = 0$ to find the equilibrium solutions.

$$y' = 0 \Leftrightarrow y^3 - 5y^2 + 6y = 0 \Leftrightarrow y(y-2)(y-3) = 0 \Leftrightarrow y = 0, y = 2 \text{ or } y = 3.$$

To determine if the equilibria are stable or not, you need to know when $y' < 0$ and when $y' > 0$:

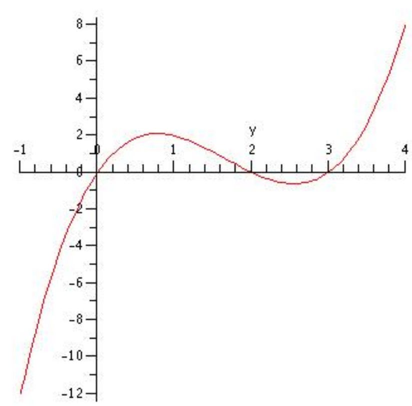
$$y' < 0 \Leftrightarrow y < 0 \text{ or } 2 < y < 3 \text{ and } y' > 0 \Leftrightarrow 0 < y < 2 \text{ or } y > 3.$$

See the graph to the right, which shows y' as a function of y. Then

$y = 0$ **is unstable:** if $y < 0$ then $y' < 0$, so y will decrease farther from $y = 0$; if $y > 0$ then $y' > 0$, so y will increase farther from $y = 0$.

$y = 2$ **is stable:** if $y < 2$ then $y' > 0$, so y will increase closer to $y = 2$; if $y > 2$ then $y' < 0$, so y will decrease closer to $y = 2$.

$y = 3$ **is unstable:** if $y < 3$ then $y' < 0$, so y will decrease farther from $y = 3$; if $y > 3$ then $y' > 0$, so y will increase farther from $y = 3$.



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16 [10 marks; avg: 4.01/10] As the salt KNO_3 dissolves in methanol, the number $x(t)$ of grams of the salt in solution after t seconds satisfies the differential equation

$$\frac{dx}{dt} = 0.6\,x - 0.003\,x^2.$$

(a) [6 marks] If $x = 50$ when $t = 0$, how long will it take for an additional 50 g of salt to dissolve?

Solution: this is the DE for logistic growth.

$$\frac{dx}{dt} = 0.6\,x - 0.003\,x^2 \Leftrightarrow \frac{dx}{dt} = 0.6\,x\left(1 - \frac{x}{200}\right),$$

which has solution

$$x = \frac{200}{1 + A\,e^{-0.6\,t}},$$

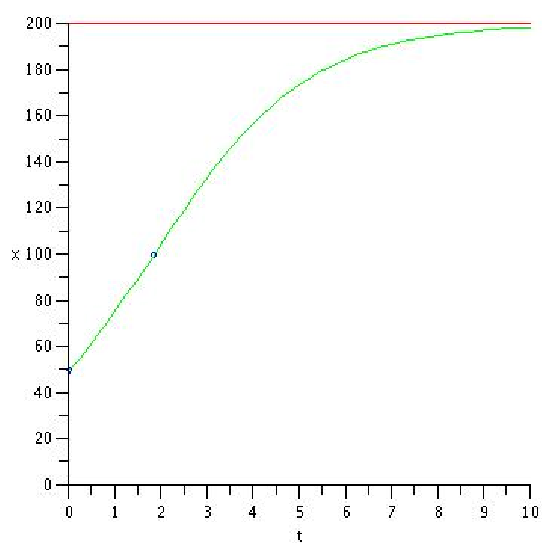
for some constant A. To find A, let $t = 0$ and $x = 50$: $50 = 200/(1 + A) \Rightarrow A = 3$. Finally, let $x = 100$ and solve for t:

$$100 = \frac{200}{1 + 3\,e^{-0.6\,t}} \Leftrightarrow 1 + 3\,e^{-0.6\,t} = 2 \Leftrightarrow 3 = e^{0.6\,t} \Leftrightarrow t = \frac{5\ln 3}{3} \approx 1.831$$

So it will take approximately 1.8 sec for 50 more grams of salt to dissolve.

(b) [4 marks] Sketch the graph of x for $t \geq 0$, indicating any inflection points and horizontal asymptotes.

Graph: your graph must pass through the two indicated points, $(0, 50)$ and $(1.8, 100)$, the second of which is the inflection point, be asymptotic to $x = 200$, and have the correct concavity.



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This page is for rough work; it will not be marked unless you indicate to mark it.

The end.