University of Toronto
MAT 187H1F TERM TEST
WEDNESDAY, MAY 21, 2014
Duration: 100 minutes

Only aids permitted: Casio FX-991 or Sharp EL-520 calculator.

Instructions: Present your solutions to the following questions in the booklets provided. The value for each question is indicated in parantheses beside the question number. You may find the sheet of formulas on the back of this page useful. **TOTAL MARKS: 60**

Answers:

1. Find the exact values of the following integrals:

   (a) [4 marks] $\displaystyle\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx = \frac{2}{35}$

   (b) [4 marks] $\displaystyle\int_{-\ln\sqrt{3}}^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\, dx = \frac{\pi}{6}$

2. [8 marks] Find the exact value of $\displaystyle\int_1^{\sqrt{2}} x \sec^{-1} x\, dx = \frac{\pi}{4} - \frac{1}{2}$

3. [8 marks] Find $\displaystyle\int \frac{5x^3+5x^2+x-3}{(x+1)^2(x^2+1)}\, dx = \frac{2}{x+1} + \ln|x+1| - 2\tan^{-1} x + 2\ln(x^2+1) + C$

4. [8 marks] Find the exact value of $\displaystyle\int_0^2 \frac{x^2}{(4+x^2)^2}\, dx = \frac{\pi}{16} - \frac{1}{8}$

5. [8 marks] Find $\displaystyle\int \frac{(3-x)\, dx}{\sqrt{5+4x-x^2}} = \sin^{-1}\left(\frac{x-2}{3}\right) + \sqrt{5+4x-x^2} + C$

6. Find the exact values of the following improper integrals:

   (a) [4 marks] $\displaystyle\int_1^{\infty} \frac{1}{x^2}\, dx = 1$

   (b) [4 marks] $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx = 2$

7. [6 marks] Find $\displaystyle\int \frac{1}{x\sqrt{x+1}}\, dx = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

8. [6 marks] Find $\displaystyle\int \sin\sqrt{x}\, dx = 2\sin\sqrt{x} - 2\sqrt{x}\cos\sqrt{x} + C$

Solutions:

1.(a) [4 marks] Let $u = \sin x$, then $du = \cos x\,dx$ and $\cos^2 x = 1 - \sin^2 x = 1 - u^2$. So

$$\int_0^{\pi/2} \sin^4 x\, \cos^3 x\, dx\, dx = \int_0^1 u^4(1-u^2)\,du = \int_0^1 (u^4 - u^6)\,du = \left[\frac{u^5}{5} - \frac{u^7}{7}\right]_0^1 = \frac{2}{35}$$

1.(b) [4 marks] Let $u = e^{2x}$, then $du = e^x\,dx$ and

$$\int_{-\ln\sqrt{3}}^{\ln\sqrt{3}} \frac{e^x}{1+e^{2x}}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{1+u^2}\,du = \left[\tan^{-1} u\right]_{1/\sqrt{3}}^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

2 [8 marks] Use parts: let $u = \sec^{-1} x$ and let $dv = x\,dx$. Then

$$du = \frac{1}{x\sqrt{x^2-1}}\,dx \text{ and } v = \int x\,dx = \frac{x^2}{2},$$

so

$$\begin{aligned}
\int_1^{\sqrt{2}} x\,\sec^{-1} x\,dx = \int_1^{\sqrt{2}} u\,dv &= [uv]_1^{\sqrt{2}} - \int_1^{\sqrt{2}} v\,du \\
&= \left[\frac{x^2}{2}\sec^{-1} x\right]_1^{\sqrt{2}} - \frac{1}{2}\int_1^{\sqrt{2}} \frac{x^2}{x\sqrt{x^2-1}}\,dx \\
&= \sec^{-1}\sqrt{2} - \frac{1}{2}\sec^{-1} 1 - \frac{1}{2}\int_1^{\sqrt{2}} \frac{x}{\sqrt{x^2-1}}\,dx \\
&= \frac{\pi}{4} - 0 - \frac{1}{2}\left[\sqrt{x^2-1}\right]_1^{\sqrt{2}} \\
&= \frac{\pi}{4} - \frac{1}{2}
\end{aligned}$$

3. [8 marks] Use partial fractions.

$$\begin{aligned}
\frac{5x^3+5x^2+x-3}{(x+1)^2(x^2+1)} &= \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1} \\
&= \frac{A(x+1)(x^2+1) + B(x^2+1) + (Cx+D)(x+1)^2}{(x+1)^2(x^2+1)} \\
&= \frac{(A+C)x^3 + (A+B+2C+D)x^2 + (A+C+2D)x + A+B+D}{x^4-3x^2-4}
\end{aligned}$$

$$\text{Then} \begin{cases} A + C = 5 \\ A + B + 2C + D = 5 \\ A + C + 2D = 1 \\ A + B + D = -3 \end{cases} \Rightarrow A = 1, B = -2, C = 4, D = -2.$$

So

$$\begin{aligned}
\int \frac{5x^3+5x^2+x-3}{(x+1)^2(x^2+1)}\,dx &= \int \frac{dx}{x+1} - \int \frac{2\,dx}{(x+1)^2} + \int \frac{4x\,dx}{x^2+1} - \int \frac{2\,dx}{x^2+1} \\
&= \ln|x+1| + \frac{2}{x+1} + 2\ln(x^2+1) - 2\tan^{-1} x + K
\end{aligned}$$

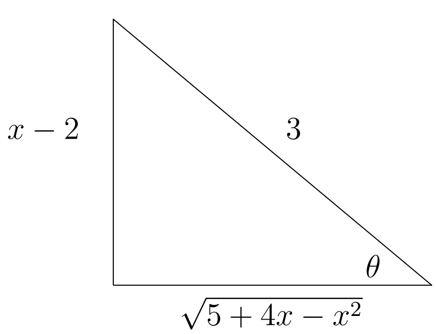
4. [8 marks] Use a trig substitution. Let $x = 2\tan\theta$; then $dx = 2\sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int_0^2 \frac{x^2}{(4+x^2)^2}\, dx &= \int_0^{\pi/4} \frac{4\tan^2\theta}{(4\sec^2\theta)^2}\, 2\sec^2\theta\, d\theta \\
&= \frac{1}{2}\int_0^{\pi/4} \sin^2\theta\, d\theta \\
&= \frac{1}{2}\int_0^{\pi/4} \left(\frac{1-\cos(2\theta)}{2}\right) d\theta \\
&= \frac{1}{4}\left[\theta - \frac{\sin(2\theta)}{2}\right]_0^{\pi/4} \\
&= \frac{1}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right) \\
&= \frac{\pi}{16} - \frac{1}{8}
\end{aligned}$$

5. [8 marks] Complete the square and use a trig substitution:

$$5 + 4x - x^2 = 9 - 4 + 4x - x^2 = 9 - (x-2)^2,$$

so let $x - 2 = 3\sin\theta$. Then $x = 2 + 3\sin\theta$ and $dx = 3\cos\theta\, d\theta$. Consequently



$$\begin{aligned}
&\int \frac{(3-x)\, dx}{\sqrt{5+4x-x^2}} \\
=\ &\int \frac{(1-3\sin\theta)}{\sqrt{9-9\cos^2\theta}}\, 3\cos\theta\, d\theta \\
=\ &\int (1-3\sin\theta)\, d\theta \\
=\ &\theta + 3\cos\theta + C \\
=\ &\sin^{-1}\left(\frac{x-2}{3}\right) + 3\left(\frac{\sqrt{5+4x-x^2}}{3}\right) + C \\
=\ &\sin^{-1}\left(\frac{x-2}{3}\right) + \sqrt{5+4x-x^2} + C
\end{aligned}$$

6(a) [4 marks] This is an improper integral.

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b dx = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 0+1 = 1$$

6(b) [4 marks] This is also an improper integral.

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to a^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+}\left[2\sqrt{x}\right]_a^1 = \lim_{a\to 0^+}\left(2-2\sqrt{a}\right) = 2-0 = 2$$

7. [6 marks] Let $x+1=u^2$. Then $dx = 2u\,du$ and $x = u^2-1$, so that

$$\int \frac{1}{x\sqrt{x+1}}\,dx = \int \frac{2u\,du}{(u^2-1)\,u} = \int \frac{2\,du}{u^2-1} = \int \frac{2\,du}{(u-1)(u+1)} = \int\left(\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$$= \ln|u-1| - \ln|u+1| + C = \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$$

8. [6 marks] Let $x = t^2$ and then use integration by parts.

$$\begin{aligned}
\int \sin\sqrt{x}\,dx &= \int 2t\,\sin t\,dt \\
(\text{let } u = 2t, dv = \sin t\,dt) &= \int u\,dv \\
&= uv - \int v\,du \\
&= 2t(-\cos t) - \int(-2\cos t)\,dt \\
&= -2t\cos t + 2\sin t + C \\
&= -2\sqrt{x}\,\cos\sqrt{x} + 2\,\sin\sqrt{x} + C
\end{aligned}$$