University of Toronto
Solutions to **MAT 187H1F TERM TEST**
of **THURSDAY, MAY 17, 2012**
Duration: 110 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.
Answers:

1. [5 marks] $\displaystyle\int_0^{\pi/2} \sin^4 x\, \cos^3 x\, dx = \frac{2}{35}$

2. [5 marks] $\displaystyle\int_0^1 \frac{3+6x}{1+(3x+3x^2)^2}\, dx = \tan^{-1} 6$

3. [8 marks] $\displaystyle\int \sqrt{x}\, \sec^{-1}\sqrt{x+1}\, dx = \frac{2}{3}x^{3/2}\sec^{-1}\sqrt{x+1} - \frac{x}{3} + \frac{\ln|x+1|}{3} + C$

4. [10 marks] $\displaystyle\int \frac{3x^3-3x^2+9x-18}{x^4+9x^2}\, dx = \ln|x| + \frac{2}{x} + \ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

5. [8 marks] $\displaystyle\int_0^1 \frac{x^3\, dx}{\sqrt{1+x^2}} = \frac{2-\sqrt{2}}{3}$

6. [8 marks] $\displaystyle\int \frac{x\, dx}{\sqrt{4x-x^2}} = 2\sin^{-1}\left(\frac{x-2}{2}\right) - \sqrt{4x-x^2} + C$

7. [4 marks] $\displaystyle\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \frac{\pi}{2}$

8. [5 marks] $\displaystyle\int_0^1 \frac{x^2\, dx}{\sqrt{1-x^2}} = \frac{\pi}{4}$

9. [7 marks] $\displaystyle\int e^x \cos x\, dx = \frac{e^x\sin x + e^x\cos x}{2} + C$

Solutions:

1. Let $u = \sin x$, then $du = \cos x\, dx$ and $\cos^2 x = 1 - \sin^2 x = 1 - u^2$. So

$$\int_0^{\pi/2} \sin^4 x\, \cos^3 x\, dx = \int_0^1 u^4(1-u^2)\, du = \int_0^1 (u^4 - u^6)\, du = \left[\frac{u^5}{5} - \frac{u^7}{7}\right]_0^1 = \frac{2}{35}$$

2. Let $u = 3x + 3x^2$, then $du = (3 + 6x)\, dx$ and

$$\int_0^1 \frac{3+6x}{1+(3x+3x^2)^2}\, dx = \int_0^6 \frac{du}{1+u^2} = \left[\tan^{-1} u\right]_0^6 = \tan^{-1} 6$$

3. Use parts: let $u = \sec^{-1}\sqrt{x+1}$ and let $dv = \sqrt{x}$. Then

$$du = \frac{1}{\sqrt{x+1}}\,\frac{1}{\sqrt{x+1-1}}\,\frac{1}{2\sqrt{x+1}}\, dx = \frac{dx}{2(x+1)\sqrt{x}};\;\; v = \int \sqrt{x}\, dx = \frac{2}{3}x^{3/2},$$

so

$$\begin{aligned}\int \sqrt{x}\,\sec^{-1}\sqrt{x+1}\, dx = \int u\, dv &= uv - \int v\, du \\ &= \frac{2}{3}x^{3/2}\sec^{-1}\sqrt{x+1} - \frac{1}{3}\int \frac{x\, dx}{x+1} \\ &= \frac{2}{3}x^{3/2}\sec^{-1}\sqrt{x+1} - \frac{1}{3}\int \left(1 - \frac{1}{1+x}\right) dx \\ &= \frac{2}{3}x^{3/2}\sec^{-1}\sqrt{x+1} - \frac{x}{3} + \frac{1}{3}\ln|1+x| + C\end{aligned}$$

4. Use partial fractions. Let

$$\frac{3x^3 - 3x^2 + 9x - 18}{x^4 + 9x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 9} = \frac{(Ax+b)(x^2+9) + (Cx+D)x^2}{x^2(x^2+9}$$

$$\Rightarrow A + C = 3, B + D = -3, 9A = 9, 9B = -18$$
$$\Rightarrow A = 1, B = -2, C = 2, D = -1.$$

Then

$$\begin{aligned}\int \frac{3x^3 - 3x^2 + 9x - 18}{x^4 + 9x^2}\, dx &= \int \frac{dx}{x} - \frac{2\, dx}{x^2} + \frac{2x\, dx}{x^2+9} - \frac{dx}{x^2+9} \\ &= \ln|x| + \frac{2}{x} + \ln(x^2+9) - \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C\end{aligned}$$

5. There are lots of different ways to do this question. Here are two possible ways:

Method 1: use a trig substitution. Let $x = \tan\theta$; then $dx = \sec^2\theta\, d\theta$ and

$$\int_0^1 \frac{x^3\, dx}{\sqrt{1+x^2}} = \int_0^{\pi/4} \frac{\tan^3\theta}{\sec\theta} \sec^2\theta\, d\theta = \int_0^{\pi/4} (\sec^2\theta - 1)\, \tan\theta \sec\theta\, d\theta.$$

Now let $u = \sec\theta$:

$$\int_0^{\pi/4} (\sec^2\theta - 1)\, \tan\theta \sec\theta\, d\theta = \int_1^{\sqrt{2}} (u^2 - 1)\, du = \left[\frac{u^3}{3} - u\right]_1^{\sqrt{2}} = \frac{2-\sqrt{2}}{3}.$$

Method 2: use the substitution $u^2 = 1 + x^2$; then $2u\, du = 2x\, dx$ and

$$\int_0^1 \frac{x^3\, dx}{\sqrt{1+x^2}} = \int_0^1 \frac{x^2\, x dx}{\sqrt{1+x^2}} = \int_1^{\sqrt{2}} \frac{u^2-1}{u}\, u du = \int_1^{\sqrt{2}} (u^2 - 1)\, du = \frac{2-\sqrt{2}}{3},$$

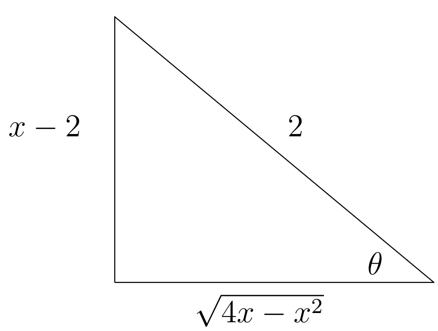
as before.

6. Complete the square and use a trig substitution:

$$4x - x^2 = 4 - 4 + 4x - x^2 = 4 - (x-2)^2 = 2^2 - (x-2)^2,$$

so let $x - 2 = 2\sin\theta$. Then $x = 2 + 2\sin\theta$ and $dx = 2\cos\theta$. Then

$$\int \frac{x\, dx}{\sqrt{4x - x^2}} = \int \frac{(2 + 2\sin\theta)}{\sqrt{4 - 4\sin^2\theta}} 2\cos\theta\, d\theta = \int (2 + 2\sin\theta)\, d\theta = 2\theta - 2\cos\theta + C$$



In the triangle to the left,

$$x - 2 = 2\sin\theta \Leftrightarrow \sin\theta = \frac{x-2}{2}.$$

So

$$\cos\theta = \frac{\sqrt{4x - x^2}}{2}.$$

Thus

$$\int \frac{x\, dx}{\sqrt{4x - x^2}} = 2\theta - 2\cos\theta + C = 2\sin^{-1}\left(\frac{x-2}{2}\right) - \sqrt{4x - x^2} + C.$$

7. This is an improper integral.

$$\int_1^\infty \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to\infty}\int_1^b \frac{dx}{x\sqrt{x^2-1}} = \lim_{b\to\infty}\left[\sec^{-1} x\right]_1^b = \lim_{b\to\infty} \sec^{-1} b - \sec^{-1} 1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

8. This is also an improper integral, but after the trig substitution $x = \sin\theta$ everything is proper:

$$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^2}} = \int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta = \int_0^{\pi/2}\sin^2\theta\,d\theta.$$

Now use a double angle formula:

$$\int_0^{\pi/2}\sin^2\theta\,d\theta = \int_0^{\pi/2}\frac{1-\cos(2\theta)}{2}\,d\theta = \left[\frac{\theta}{2} - \frac{\sin(2\theta)}{4}\right]_0^{\pi/2} = \frac{\pi}{4}.$$

9. Use integration by parts, twice.

Start with $u = e^x, dv = \cos x\,dx$. Then $du = e^x\,dx, v = \sin x$ and

$$\int e^x\cos x\,dx = \int u\,dv = uv - \int v\,du = e^x\sin x - \int e^x \sin x\,dx.$$

To integrate $\int e^x \sin x\,dx$ let $s = e^x, dt = \sin x\,dx$. Then $ds = e^x\,dx, t = -\cos x$ and

$$\int e^x\sin x\,dx = \int s\,dt = st - \int t\,ds = -e^x\cos x + \int e^x\cos x\,dx.$$

Putting it all together,

$$\int e^x\cos x\,dx = e^x\sin x + e^x\cos x - \int e^x\cos x\,dx$$

$$\Rightarrow 2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x + C$$

$$\Rightarrow \int e^x\cos x\,dx = \frac{e^x\sin x + e^x\cos x}{2} + C.$$