University of Toronto
Solutions to **MAT 187H1F TERM TEST**
of **THURSDAY, MAY 19, 2011**
Duration: 100 minutes

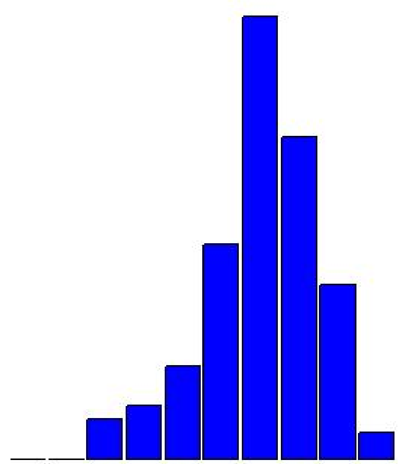
Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.
Instructions: Find the seven following integrals. Present your solutions in the space provided. The value for each question is indicated in parantheses beside the question number. TOTAL MARKS: 60

General Comments: The first five questions in this test are pretty routine. Only Question 6 was supposed to be a challenge.

1. Question #5 is just a direct application of the given formula #17.
2. Question #4 requires the use of double angle formulas.
3. Question #6 requires partial fractions, completing the square and a trig substitution, just to find the antiderivative. Then you have to use properties of logarithms to evaluate the improper integral properly. It is definitely a challenging question but the challenge is simply in executing correctly all the methods we have covered; it does not require any new or brilliant steps.

Breakdown of Results: 102 registered students wrote this test. The marks ranged from 20% to 93.3%, and the average was 64.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	2.0%
A	14.7%	80-89%	12.7%
B	23.5%	70-79%	23.5%
C	32.4%	60-69%	32.4%
D	15.7%	50-59%	15.7%
F	13.7%	40-49%	6.9%
		30-39%	3.9%
		20-29%	2.9%
		10-19%	0%
		0-9%	0%



Formulas you may find useful. Do not tear this page from the test.

1. $\int e^u\, du = e^u + C$
2. $\displaystyle\int u^n\, du = \frac{u^{n+1}}{n+1} + C, n \neq -1$
3. $\displaystyle\int \frac{1}{u}\, du = \ln|u| + C$
4. $\displaystyle\int \cos u\, du = \sin u + C$
5. $\displaystyle\int \sin u\, du = -\cos u + C$
6. $\displaystyle\int \sec^2 u\, du = \tan u + C$
7. $\displaystyle\int \sec u \tan u\, du = \sec u + C$
8. $\displaystyle\int \csc^2 u\, du = -\cot u + C$
9. $\displaystyle\int \csc u \cot u\, du = -\csc u + C$
10. $\displaystyle\int \tan u\, du = \ln|\sec u| + C$
11. $\displaystyle\int \sec u\, du = \ln|\sec u + \tan u| + C$
12. $\displaystyle\int \cot u\, du = -\ln|\csc u| + C$
13. $\displaystyle\int \csc u\, du = -\ln|\csc u + \cot u| + C$
14. $\displaystyle\int \frac{1}{\sqrt{a^2 - u^2}}\, du = \sin^{-1}\frac{u}{a} + C = \arcsin\frac{u}{a} + C$
15. $\displaystyle\int \frac{1}{a^2 + u^2}\, du = \frac{1}{a}\tan^{-1}\frac{u}{a} + C = \frac{1}{a}\arctan\frac{u}{a} + C$
16. $\displaystyle\int \frac{1}{u\sqrt{u^2 - a^2}}\, du = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C = \frac{1}{a}\text{arcsec}\left|\frac{u}{a}\right| + C$
17. $\displaystyle\int \cos^n x\, dx = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n}\int \cos^{n-2} x\, dx$
18. $\sin^2\theta + \cos^2\theta = 1$
19. $\tan^2\theta + 1 = \sec^2\theta$
20. $\sin(2\theta) = 2\sin\theta\, \cos\theta$
21. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

1.(a) [7 marks] $\int \sec^7 x \, \tan^3 x \, dx$

Soluton: let $u = \sec x$. Then $du = \sec x \, \tan x \, dx$, and

$$\begin{aligned}
\int \sec^7 x \, \tan^3 x \, dx &= \int \sec^6 x \, \tan^2 x \, \sec x \, \tan x \, dx \\
&= \int \sec^6 x \, (\sec^2 x - 1) \, \sec x \, \tan x \, dx \\
&= \int u^6 (u^2 - 1) \, du = \int (u^8 - u^6) \, du \\
&= \frac{u^9}{9} - \frac{u^7}{7} + C \\
&= \frac{1}{9} \sec^9 x - \frac{1}{7} \sec^7 x + C
\end{aligned}$$

1.(b) [7 marks] $\int \sec^{-1} \sqrt{x} \, dx$

Soluton: Use integration by parts, and be careful!

$$\begin{aligned}
\int \sec^{-1} \sqrt{x} \, dx = \int u \, dv &= uv - \int v \, du, \text{ with } u = \sec^{-1} \sqrt{x}, dv = dx \\
&= x \sec^{-1} \sqrt{x} - \int \frac{1}{\sqrt{x}\sqrt{x-1}} \frac{1}{2\sqrt{x}} \, x \, dx \\
&= x \sec^{-1} \sqrt{x} - \frac{1}{2} \int \frac{dx}{\sqrt{x-1}} \\
&= x \sec^{-1} \sqrt{x} - \sqrt{x-1} + C
\end{aligned}$$

2. [10 marks] $\displaystyle\int \frac{2x^3-2x^2+4x}{(x^2+1)(x-1)^2}\,dx$

Soluton: Let

$$\frac{2x^3-2x^2+4x}{(x^2+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Then

$$\begin{aligned} \frac{2x^3-2x^2+4x}{(x^2+1)(x-1)^2} &= \frac{A(x^2+1)(x-1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x^2+1)(x-1)^2} \\ \Leftrightarrow 2x^3-2x^2+4x &= (A+C)x^3+(B-A-2C+D)x^2+(A+C-2D)x-A+B+D \end{aligned}$$

Solve:

$$\left\{\begin{array}{rcrcrcrcr} A & & & + & C & & & = & 2 \\ -A & + & B & - & 2C & + & D & = & -2 \\ A & & & + & C & - & 2D & = & 4 \\ -A & + & B & & & + & D & = & 0 \end{array}\right. \Leftrightarrow (A,B,C,D) = (1,2,1,-1).$$

So

$$\begin{aligned} \int \frac{2x^3-2x^2+4x}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{x-1}{x^2+1}\right)dx \\ &= \int \frac{1}{x-1}\,dx + \int \frac{2}{(x-1)^2}\,dx + \frac{1}{2}\int \frac{2x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx \\ &= \ln|x-1| - \frac{2}{x+2} + \frac{1}{2}\ln(x^2+1) - \tan^{-1}x + C \end{aligned}$$

3. [8 marks] $\displaystyle\int_0^2 \frac{x\,dx}{\sqrt{2x-x^2}}$

Soluton: complete the square and use a trigonometric substitution.

$$\begin{aligned} 2x - x^2 &= -(x^2 - 2x) \\ &= -(x^2 - 2x + 1 - 1) \\ &= -\left(x^2 - 2x + 1\right) + 1 \\ &= -(x-1)^2 + 1 \\ &= 1 - (x-1)^2 \end{aligned}$$

Let

$$x - 1 = \sin\theta;$$

then, putting everything in terms of θ :

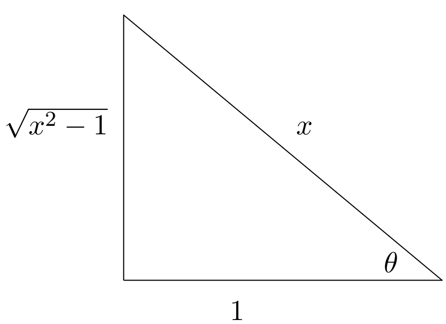
$$\begin{aligned} \int_0^2 \frac{x\,dx}{\sqrt{2x-x^2}} &= \int_{-\pi/2}^{\pi/2} \frac{1+\sin\theta}{\sqrt{1-\sin^2\theta}} \cos\theta\,d\theta \\ &= \int_{-\pi/2}^{\pi/2} (1+\sin\theta)\,d\theta \\ &= [\theta - \cos\theta]_{-\pi/2}^{\pi/2} \\ &= \pi \end{aligned}$$

Note that after the substitution the integral is no longer improper.

4. [8 marks] $\displaystyle\int \frac{dx}{x^3\sqrt{x^2-1}}$

Soluton: Let $x = \sec\theta$; then $dx = \sec\theta\tan\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{dx}{x^3\sqrt{x^2-1}} &= \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^3\theta\sqrt{\sec^2\theta-1}} \\
&= \int \cos^2\theta\, d\theta \\
&= \frac{1}{2}\int (1+\cos(2\theta))\, d\theta \\
&= \frac{1}{2}\left(\theta + \frac{\sin(2\theta)}{2}\right) + C \\
&= \frac{1}{2}\left(\theta + \frac{2\sin\theta\cos\theta}{2}\right) + C \\
&= \frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2} + C
\end{aligned}$$



In the triangle to the left, $\sec\theta = x$. From the triangle

$$\sin\theta = \frac{\sqrt{x^2-1}}{x} \text{ and } \cos\theta = \frac{1}{x}.$$

Thus

$$\int \frac{dx}{x^3\sqrt{x^2-1}} = \frac{1}{2}\sec^{-1}x + \frac{1}{2}\frac{\sqrt{x^2-1}}{x^2} + C.$$

5. [8 marks] $\displaystyle\int_0^{\pi/2} \cos^6 x\,dx$

Soluton: use the reduction formula #17 three times:

$$\begin{aligned}
\int_0^{\pi/2} \cos^6 x\,dx &= \left[\frac{\sin x\cos^5 x}{6}\right]_0^{\pi/2} + \frac{5}{6}\int_0^{\pi/2} \cos^4 x\,dx \\
&= 0 + \frac{5}{6}\left(\left[\frac{\sin x\cos^3 x}{4}\right]_0^{\pi/2} + \frac{3}{4}\int_0^{\pi/2} \cos^2 x\,dx\right) \\
&= 0 + \frac{5}{6}\cdot\frac{3}{4}\int_0^{\pi/2} \cos^2 x\,dx \\
&= \frac{5}{8}\left(\left[\frac{\sin x\cos x}{2}\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2} d\theta\right) \\
&= 0 + \frac{5}{8}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \\
&= \frac{5\pi}{32}
\end{aligned}$$

6. [12 marks] $\int_0^\infty \frac{dx}{x^3+1}$

Soluton: $x^3+1=(x+1)(x^2-x+1)$. Let

$$\frac{1}{x^3+1}=\frac{A}{x+1}+\frac{Bx+C}{x^2-x+1}.$$

Then

$$1=A(x^2-x+1)+(Bx+C)(x+1)=(A+B)x^2+(-A+B+C)x+A+C,$$

from which

$$\left\{\begin{array}{rcrcrcl} A & + & B & & & = & 0 \\ -A & + & B & + & C & = & 0 \\ A & & & + & C & = & 1 \end{array}\right. \Leftrightarrow (A,B,C)=\left(\frac{1}{3},-\frac{1}{3},\frac{2}{3}\right).$$

Hence

$$\begin{aligned}
\int \frac{dx}{x^3+1} &= \frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\int\frac{(x-2)\,dx}{x^2-x+1} \\
\text{(complete the square)} &= \frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\int\frac{(x-2)\,dx}{x^2-x+1/4+3/4} \\
&= \frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\int\frac{(x-2)\,dx}{(x-1/2)^2+3/4} \\
\left(\text{let } x-\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta\right) &= \frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\int\frac{\frac{\sqrt{3}}{2}\tan\theta-\frac{3}{2}}{\frac{3}{4}\tan^2\theta+\frac{3}{4}}\,\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta \\
&= \frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\int\tan\theta\,d\theta+\frac{1}{\sqrt{3}}\int d\theta \\
&= \frac{1}{3}\int\frac{dx}{x+1}-\frac{1}{3}\ln|\sec\theta|+\frac{1}{\sqrt{3}}\theta+C \\
\left(\sec\theta=2\sqrt{x^2-x+1}/\sqrt{3}\right) &= \frac{1}{3}\ln|x+1|-\frac{1}{3}\ln\sqrt{x^2-x+1}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+\frac{1}{3}\ln\frac{\sqrt{3}}{2}+C \\
&= \frac{1}{3}\ln\frac{|x+1|}{\sqrt{x^2-x+1}}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+C'
\end{aligned}$$

So, finally,

$$\begin{aligned}
\int_0^\infty\frac{dx}{x^3+1} &= \lim_{b\to\infty}\left[\frac{1}{3}\ln\frac{|b+1|}{\sqrt{b^2-b+1}}+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2b-1}{\sqrt{3}}\right)\right]_0^b \\
&= \frac{1}{3}(\ln 1-\ln 1)+\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}+\frac{\pi}{6}\right) \\
&= \frac{2\pi}{3\sqrt{3}}
\end{aligned}$$