University of Toronto
Solutions to **MAT 187H1F TERM TEST**
of **THURSDAY, MAY 20, 2010**
Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.
Instructions: Find the seven following integrals. Present your solutions in the booklets provided. The value for each question is indicated in parantheses beside the question number. TOTAL MARKS: 60

General Comments: The questions in this test are mostly routine. The only complications are in Questions 3 and 6.

1. In Question 3: find the antiderivative using the method of partial fractions first, then bother about the improper integral.

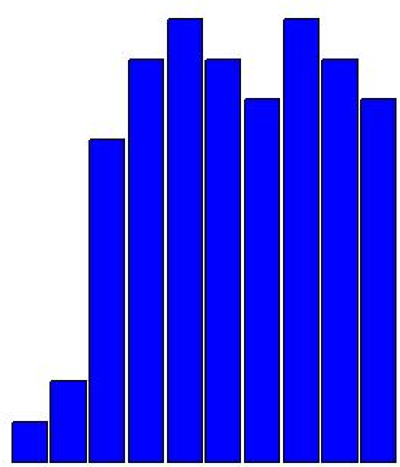
2. In Question 6: use integration by parts, twice. You could also let $z = \ln x \Leftrightarrow x = e^z$; then the integral becomes

$$\int \sin(\ln x)\, dx = \int e^z \sin z \, dz,$$

which is more obviously done by parts.

Breakdown of Results: 81 registered students wrote this test. The marks ranged from 6.7% to 96.7%, and the average was 58.2%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right. The test results include 8 students who arrived late because of the TTC delay. These 8 tests had an average of 53.3%, and may have lowered the overall course average for this test.

Grade	%	Decade	%
		90-100%	11.1%
A	23.4%	80-89%	12.3%
B	13.6%	70-79%	13.6%
C	11.1%	60-69%	11.1%
D	12.3%	50-59%	12.3%
F	39.5%	40-49%	13.6%
		30-39%	12.3%
		20-29%	9.9%
		10-19%	2.5%
		0-9%	1.2%



1. [6 marks] $\int \sec^5 x \, \tan x \, dx$

Soluton: let $u = \sec x$. Then $du = \sec x \, \tan x \, dx$, and

$$\begin{aligned}
\int \sec^5 x \, \tan x \, dx &= \int \sec^4 x \, \sec x \, \tan x \, dx \\
&= \int u^4 \, du \\
&= \frac{u^5}{5} + C \\
&= \frac{1}{5} \sec^5 x + C
\end{aligned}$$

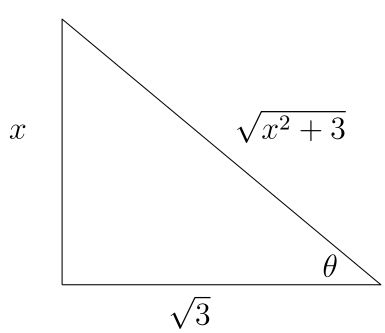
2. [10 marks] $\displaystyle\int_1^3 \frac{dx}{x^2\sqrt{x^2+3}}$

Soluton: Let $x = \sqrt{3}\tan\theta$; then $dx = \sqrt{3}\sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int_1^3 \frac{dx}{x^2\sqrt{x^2+3}} &= \int_{\pi/6}^{\pi/3} \frac{\sqrt{3}\sec^2\theta\, d\theta}{3\tan^2\theta\sqrt{3\tan^2\theta+3}} \\
&= \frac{1}{3}\int_{\pi/6}^{\pi/3} \frac{\sec\theta}{\tan^2\theta}\, d\theta \\
&= \frac{1}{3}\int_{\pi/6}^{\pi/3} \frac{\cos\theta}{\sin^2\theta}\, d\theta \\
(\text{let } w = \sin\theta) &= \frac{1}{3}\int_{1/2}^{\sqrt{3}/2} \frac{dw}{w^2} \\
&= \frac{1}{3}\left[-\frac{1}{w}\right]_{1/2}^{\sqrt{3}/2} \\
&= \frac{2}{3}\left(1 - \frac{1}{\sqrt{3}}\right)
\end{aligned}$$

NOTE: if you don't change the limits of integration as you change the variable, then you should find the indefinite integral first. As above,

$$\int \frac{dx}{x^2\sqrt{x^2+3}} = -\frac{1}{3}\left(\frac{1}{w}\right) + C = -\frac{1}{3}\csc\theta + C.$$



In the triangle to the left, $\tan\theta = \dfrac{x}{\sqrt{3}}$. So

$$\csc\theta = \frac{\sqrt{3+x^2}}{x}.$$

Then

$$\int_1^3 \frac{dx}{x^2\sqrt{x^2+3}} = \left[-\frac{1}{3}\frac{\sqrt{3+x^2}}{x}\right]_1^3 = -\frac{1}{3}\frac{\sqrt{12}}{3} + \frac{1}{3}\frac{\sqrt{4}}{1} = \frac{2}{3} - \frac{2}{3\sqrt{3}}.$$

3. [12 marks] $\displaystyle\int_0^\infty \frac{x^2+x+8}{(x^2+1)(x+2)^2}\,dx$

Soluton: First find the antiderivative using the method of partial fractions. Let

$$\frac{x^2+x+8}{(x^2+1)(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{Cx+D}{x^2+1}.$$

Then

$$\begin{aligned}
\frac{x^2+x+8}{(x^2+1)(x+2)^2} &= \frac{A(x^2+1)(x+2)+B(x^2+1)+(Cx+D)(x+2)^2}{(x^2+1)(x+2)^2} \\
\Leftrightarrow x^2+x+8 &= (A+C)x^3+(2A+B+4C+D)x^2+(A+4C+4D)x+2A+B+4D
\end{aligned}$$

Solve:

$$\left\{\begin{array}{rcrcrcrcl}
A & & & + & C & & & = & 0 \\
2A & + & B & + & 4C & + & D & = & 1 \\
A & & & + & 4C & + & 4D & = & 1 \\
2A & + & B & & & + & 4D & = & 8
\end{array}\right. \Leftrightarrow (A,B,C,D)=(1,2,-1,1).$$

So

$$\begin{aligned}
\int \frac{x^2+x+8}{(x^2+1)(x+2)^2}\,dx &= \int\left(\frac{1}{x+2}+\frac{2}{(x+2)^2}-\frac{x-1}{x^2+1}\right)dx \\
&= \int \frac{1}{x+2}\,dx + \int \frac{2}{(x+2)^2}\,dx - \int \frac{x}{x^2+1}\,dx + \int \frac{1}{x^2+1}\,dx \\
&= \ln|x+2| - \frac{2}{x+2} - \frac{1}{2}\ln(x^2+1) + \tan^{-1}x + C \\
&= \ln|x+2| - \frac{2}{x+2} - \ln\sqrt{x^2+1} + \tan^{-1}x + C \\
&= \ln\frac{|x+2|}{\sqrt{x^2+1}} - \frac{2}{x+2} + \tan^{-1}x + C
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^\infty \frac{x^2+x+8}{(x^2+1)(x+2)^2}\,dx &= \lim_{b\to\infty}\left[\ln\frac{x+2}{\sqrt{x^2+1}} - \frac{2}{x+2} + \tan^{-1}x\right]_0^b \\
&= \lim_{b\to\infty}\left(\ln\frac{b+2}{\sqrt{b^2+1}} - \frac{2}{b+2} + \tan^{-1}b\right) - \ln 2 + 1 - \tan^{-1}0 \\
&= \ln 1 - 0 + \frac{\pi}{2} - \ln 2 + 1 \\
&= 1 + \frac{\pi}{2} - \ln 2
\end{aligned}$$

4. [10 marks] $\displaystyle\int_0^4 \frac{x\,dx}{\sqrt{4x-x^2}}$

Soluton: complete the square and use a trigonometric substitution.

$$\begin{aligned} 4x - x^2 &= -(x^2 - 4x) \\ &= -(x^2 - 4x + 4 - 4) \\ &= -\left(x^2 - 4x + 4\right) + 4 \\ &= -\left(x - 2\right)^2 + 4 \\ &= 2^2 - \left(x-2\right)^2 \end{aligned}$$

Let

$$x - 2 = 2\sin\theta;$$

then, putting everything in terms of θ :

$$\begin{aligned} \int_0^4 \frac{x\,dx}{\sqrt{4x-x^2}} &= \int_{-\pi/2}^{\pi/2} \frac{2+2\sin\theta}{\sqrt{2^2 - 2^2\sin^2\theta}}\, 2\cos\theta\, d\theta \\ &= \int_{-\pi/2}^{\pi/2} (2 + 2\sin\theta)\, d\theta \\ &= \left[2\theta - 2\cos\theta\right]_{-\pi/2}^{\pi/2} \\ &= 2\pi \end{aligned}$$

Note that after the substitution the integral is no longer improper.

5. [6 marks] $\displaystyle\int \frac{x^2\,dx}{\sqrt{x-1}}$

Soluton: let $u^2 = x - 1$. Then $x = 1 + u^2$ and $dx = 2u\,du$. So:

$$\begin{aligned}
\int \frac{x^2\,dx}{\sqrt{x-1}} &= \int \frac{(1+u^2)^2}{u}\,2u\,du \\
&= 2\int \left(1 + 2u^2 + u^4\right)\,du \\
&= 2\left(u + \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + C \\
\text{(since } u = \sqrt{x-1}) &= 2\sqrt{x-1} + \frac{4}{3}(x-1)^{3/2} + \frac{2}{5}(x-1)^{5/2} + C \\
\text{(optionally)} &= \frac{2}{15}\left(8 + 4x + 3x^2\right)\sqrt{x-1} + C
\end{aligned}$$

6. [10 marks] $\int \sin(\ln x)\,dx$

Soluton: use integration by parts twice.

$$
\begin{aligned}
\int \sin(\ln x)\,dx &= \int u\,dv, \text{ with } u = \sin(\ln x), dv = dx \\
&= uv - \int v\,du \\
&= x\sin(\ln x) - \int x\,\frac{\cos(\ln x)}{x}\,dx \\
&= x\sin(\ln x) - \int \cos(\ln x)\,dx \\
&= x\sin(\ln x) - \int s\,dt, \text{ with } s = \cos(\ln x), dt = dx \\
&= x\sin(\ln x) - \left(st - \int t\,ds\right) \\
&= x\sin(\ln x) - x\cos(\ln x) + \int x\left(-\frac{\sin(\ln x)}{x}\right)\,dx \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\,dx \\
\Rightarrow 2\int \sin(\ln x)\,dx &= x\sin(\ln x) - x\cos(\ln x) + C \\
\Rightarrow \int \sin(\ln x)\,dx &= \frac{x}{2}\left(\sin(\ln x) - \cos(\ln x)\right) + C'
\end{aligned}
$$

7. [6 marks] $\int \sec^{-1}\sqrt{x}\,dx$

Soluton: Use integration by parts, and be careful!

$$\begin{aligned}
\int \sec^{-1}\sqrt{x}\,dx = \int u\,dv &= uv - \int v\,du, \text{ with } u = \sec^{-1}\sqrt{x}, dv = dx \\
&= x\sec^{-1}\sqrt{x} - \int \frac{1}{\sqrt{x}\sqrt{x-1}}\,\frac{1}{2\sqrt{x}}\,x\,dx \\
&= x\sec^{-1}\sqrt{x} - \frac{1}{2}\int \frac{dx}{\sqrt{x-1}} \\
&= x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C
\end{aligned}$$