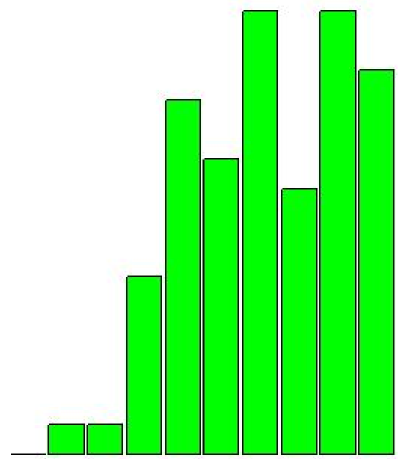
University of Toronto
Solutions to **MAT 187H1F TERM TEST**
of **WEDNESDAY, MAY 20, 2009**
Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

General Comments: All questions are routine, except for Question 7. Some of the questions were right out of the homework.

Breakdown of Results: 82 registered students wrote this test. The marks ranged from 16.7% to 100%, and the average was 66.7%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	15.9%
A	34.2%	80-89%	18.3%
B	11.0%	70-79%	11.0%
C	18.3%	60-69%	18.3%
D	12.2%	50-59%	12.2 %
F	24.3%	40-49%	14.6%
		30-39%	7.3%
		20-29%	1.2%
		10-19%	1.2%
		0-9%	0.0%



Instructions: Find the seven following integrals. Present your solutions in the booklets provided. The value for each question is indicated in parantheses beside the question number.

1. [8 marks] $\int_0^{\pi/3} \tan x \sec^4 x \, dx$

Soluton: let $u = \sec x$. Then $du = \sec x \tan x \, dx$, and

$$\begin{aligned}\int_0^{\pi/3} \tan x \sec^4 x \, dx &= \int_0^{\pi/3} \sec^3 x \sec x \tan x \, dx \\ &= \int_1^2 u^3 \, du \\ &= \left[\frac{u^4}{4}\right]_1^2 \\ &= \frac{16}{4} - \frac{1}{4} \\ &= \frac{15}{4}\end{aligned}$$

Note: you must change the limits when you change the variable.

2. [10 marks] $\displaystyle\int \frac{6x^3-12x^2+8x-4}{(x^2+1)(x-1)^2}\,dx$

Soluton: let

$$\frac{6x^3-12x^2+8x-4}{(x^2+1)(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

Then

$$\begin{aligned}
\frac{6x^3-12x^2+8x-4}{(x^2+1)(x-1)^2} &= \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1} \\
&= \frac{A(x^2+1)(x-1)+B(x^2+1)+(Cx+D)(x-1)^2}{(x-1)^2(x^2+1)} \\
\Leftrightarrow 6x^3-12x^2+8x-4 &= (A+C)x^3+(-A+B-2C+D)x^2+(A+C-2D)x-A+B+D
\end{aligned}$$

Solve

$$\left\{\begin{array}{rcrcrcrcr}
A & & & + & C & & & = & 6 \\
-A & + & B & - & 2C & + & D & = & -12 \\
A & & & + & C & - & 2D & = & 8 \\
-A & + & B & & & + & D & = & -4
\end{array}\right. \Leftrightarrow (A,B,C,D)=(2,-1,4,-1).$$

So

$$\begin{aligned}
\int \frac{6x^3-12x^2+8x-4}{(x^2+1)(x-1)^2}\,dx &= \int\left(\frac{2}{x-1} - \frac{1}{(x-1)^2} + \frac{4x-1}{x^2+1}\right)dx \\
&= 2\int\frac{1}{x-1}\,dx - \int\frac{1}{(x-1)^2}\,dx + 2\int\frac{2x}{x^2+1}\,dx - \int\frac{1}{x^2+1}\,dx \\
&= 2\ln|x-1| + \frac{1}{x-1} + 2\ln(x^2+1) - \tan^{-1}x + C
\end{aligned}$$

3. [10 marks] $\displaystyle\int_0^1 \frac{x^2}{(4-x^2)^{3/2}}\,dx$

Soluton: Let $x = 2\sin\theta$; then $dx = 2\cos\theta\,d\theta$ and

$$\begin{aligned}
\int_0^1 \frac{x^2}{(4-x^2)^{3/2}}\,dx &= \int_0^{\pi/6} \frac{4\sin^2\theta}{(4-4\sin^2\theta)^{3/2}}\,2\cos\theta\,d\theta \\
&= \int_0^{\pi/6} \frac{\sin^2\theta}{\cos^3\theta}\,\cos\theta\,d\theta \\
&= \int_0^{\pi/6} \tan^2\theta\,d\theta \\
&= \int_0^{\pi/6} (\sec^2\theta - 1)\,d\theta \\
&= [\tan\theta - \theta]_0^{\pi/6} \\
&= \frac{1}{\sqrt{3}} - \frac{\pi}{6}
\end{aligned}$$

Again: you should change the limits of integration as you change the variable.

4. [10 marks] Find $\displaystyle\int \frac{x+4}{\sqrt{6x-x^2}}\,dx$

Soluton: complete the square and use a trigonometric substitution.

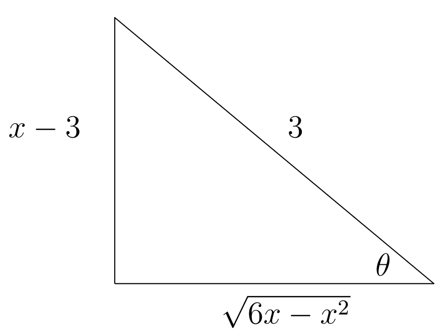
$$\begin{aligned} 6x - x^2 &= -(x^2-6x) \\ &= -(x^2-6x+9-9) \\ &= -\left(x^2-6x+9\right)+9 \\ &= -\left(x-3\right)^2+9 \\ &= 3^2-\left(x-3\right)^2 \end{aligned}$$

Let

$$x-3 = 3\sin\theta;$$

then, putting everything in terms of θ :

$$\begin{aligned} \int \frac{x+4}{\sqrt{6x-x^2}}\,dx &= \int \frac{x+4}{\sqrt{3^2-(x-3)^2}}\,dx \\ &= \int \frac{3\sin\theta+3+4}{\sqrt{3^2-3^2\sin^2\theta}}\,3\cos\theta\,d\theta \\ &= \int \frac{3\sin\theta+7}{\sqrt{\cos^2\theta}}\,\cos\theta\,d\theta \\ &= \int (3\sin\theta+7)\,d\theta \\ &= -3\cos\theta+7\theta+C \end{aligned}$$



In the triangle, $\sin\theta = \dfrac{x-3}{3}$. Then

$$\cos\theta = \frac{\sqrt{6x-x^2}}{3}.$$

So

$$\begin{aligned} \int \frac{x+4}{\sqrt{6x-x^2}}\,dx\,dx &= -3\cos\theta+7\theta+C \\ &= -3\frac{\sqrt{6x-x^2}}{3}+7\sin^{-1}\left(\frac{x-3}{3}\right)+C \\ &= -\sqrt{6x-x^2}+7\sin^{-1}\left(\frac{x-3}{3}\right)+C \end{aligned}$$

5. [8 marks] $\int \sec^{-1}\sqrt{x}\,dx$

Soluton: Use integration by parts, and be careful!

$$\begin{aligned}
\int \sec^{-1}\sqrt{x}\,dx = \int u\,dv &= uv - \int v\,du, \text{ with } u = \sec^{-1}\sqrt{x}, dv = dx \\
&= x\sec^{-1}\sqrt{x} - \int \frac{1}{\sqrt{x}\sqrt{x-1}}\,\frac{1}{2\sqrt{x}}\,x\,dx \\
&= x\sec^{-1}\sqrt{x} - \frac{1}{2}\int \frac{1}{\sqrt{x-1}}\,dx \\
&= x\sec^{-1}\sqrt{x} - \sqrt{x-1} + C
\end{aligned}$$

6. [6 marks] $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^3}\,dx$

Soluton: let $u = \ln x$.

$$
\begin{aligned}
\int_e^\infty \frac{1}{x(\ln x)^3}\,dx &= \lim_{b\to\infty}\int_e^b \frac{1}{x(\ln x)^3}\,dx \\
&= \lim_{b\to\infty}\int_1^{\ln b} \frac{1}{u^3}\,du \\
&= \lim_{b\to\infty}\left[-\frac{1}{2u^2}\right]_1^{\ln b} \\
&= \lim_{b\to\infty}\left(-\frac{1}{2(\ln b)^2}+\frac{1}{2}\right) \\
&= 0+\frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}
$$

7. [8 marks] $\displaystyle\int \frac{\sqrt{x^{2/3}-1}}{x}\,dx$

Solution: let $u^2 = x^{2/3} - 1$. Then $2u\,du = \dfrac{2}{3x^{1/3}}\,dx \Leftrightarrow u\,du = \dfrac{1}{3x^{1/3}}\,dx$, and

$$\begin{aligned}
\int \frac{\sqrt{x^{2/3}-1}}{x}\,dx &= 3\int \frac{1}{x^{2/3}}\frac{\sqrt{x^{2/3}-1}}{3x^{1/3}}\,dx \\
&= 3\int \frac{1}{u^2+1}\cdot u\cdot u\,du \\
&= 3\int \frac{u^2}{u^2+1}\,du \\
&= 3\int \left(1-\frac{1}{u^2+1}\right)\,du \\
&= 3u - 3\tan^{-1}u + C \\
&= 3\sqrt{x^{2/3}-1} - 3\tan^{-1}\sqrt{x^{2/3}-1} + C
\end{aligned}$$

Alternate Solution: let $x = \sec^3\theta$; then $dx = 3\sec^3\theta\tan\theta\,d\theta$ and

$$\begin{aligned}
\int \frac{\sqrt{x^{2/3}-1}}{x}\,dx &= \int \frac{\sqrt{\sec^2\theta-1}}{\sec^3\theta}\,3\sec^3\theta\tan\theta\,d\theta \\
&= 3\int \tan\theta\cdot\tan\theta\,d\theta \\
&= 3\int \tan^2\theta\,d\theta \\
&= 3\int \left(\sec^2\theta-1\right)\,d\theta \\
&= 3\tan\theta - 3\,\theta + C \\
&= 3\sqrt{x^{2/3}-1} - 3\sec^{-1}x^{1/3} + C
\end{aligned}$$