MAT187 - Calculus II - Winter 2016

Term Test 2 - March 8, 2016

Time allotted: 90 minutes. Aids permitted: None.

Total marks: 50

SOLUTIONS

Full Name: ______________________________
Last First

Student Number: ______________________________

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Instructions

- DO NOT WRITE ON THE QR CODE AT THE TOP OF THE PAGES.
- Please have your **student card** ready for inspection.
- This test contains 14 pages and a detached **formula sheet**. Make sure you have all of them. DO NOT DETACH ANY PAGE.
- You can use pages 13–14 to complete questions (**mark clearly** which questions you are answering).
- Calculators, cellphones, or any other electronic gadgets are not allowed. If you have a cellphone with you, it must be turned off and in a bag underneath your chair.
- DO NOT start the test until instructed to do so.

GOOD LUCK!

PART I

(10 marks)

1. (1 mark) Circle one choice. The series

$$\sum_{n=1}^{\infty} \sin(n)$$

(a) converges absolutely.

(b) converges but not absolutely.

(c) diverges.

2. (2 marks) The series

$$\sum_{n=42}^{\infty} \frac{2^{n+2}}{3^n}$$

converges to $\frac{2^{44}}{3^{41}}$ or $4 \cdot \frac{1}{1-2/3} - \sum_{n=0}^{41} \frac{2^{n+2}}{3^n}$.

3. (2 marks) The third order Taylor polynomial for $f(x) = \sin(2x)$ centred at $x = 0$ is

$p_3(x) =$ $2x - \frac{(2x)^3}{3!} = 2x - \frac{4x^3}{3}$.

4. (1 mark) Circle one choice. The series

$$\sum_{n=0}^{\infty} \frac{n}{2^n}$$

(a) is absolutely convergent.

(b) is conditionally convergent.

(c) is divergent.

5. (2 marks) Consider the function $f(x)$ with the Taylor polynomial

$$\sum_{n=0}^{10\,000} a_n(x-1)^n \qquad \text{where} \qquad a_n = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is odd} \\ n+1 & \text{if } n \text{ is even} \end{cases}$$

Then

$f^{(2016)}(1) =$ $(2016)!\, a_{2016} = (2016)! \cdot 2017 = 2017!$

6. (1 mark) Consider the function $y(t) = e^{t^2}$ and suppose that it is a solution to the differential equation

$$\frac{dy}{dt} + p(t)y = 4te^{t^2},$$

where $\boldsymbol{p(t) \neq 0}$ for all $t > 0$. Write a possible function $p(t)$:

$p(t) =$ $2t$

$\frac{dy}{dt} = 2te^{t^2}$. $\therefore$ $2te^{t^2} + p(t)\cdot e^{t^2} = 4te^{t^2}$

$p(t)\cdot \cancel{e^{t^2}} = 2t\cancel{e^{t^2}}$

$p(t) = 2t$

7. (1 mark) Let $y(t)$ be the solution to the differential equation

$$\frac{dy}{dt} = y^2 + t \qquad \text{and} \qquad y(0) = -10.$$

At how many points does the solution $y(t)$ cross the (horizontal) t-axis for $t > 0$?

one. (the function is increasing for $t>0$)

(you don't need to solve the differential equation)

PART II **Justify** your answers for all questions in this part.

8. Suppose that the temperature $u(t)$ of a cup of coffee is modelled by Newton's law of cooling **(10 marks)**

$$\frac{du}{dt} = k(u - T_0),$$

where k is a constant and $T_0 = 20^oC$ is the temperature of the room. The temperature of the coffee is 95^oC initially and after 1 minute it has cooled to 70^oC.

Determine when the coffee reaches a temperature of 45^oC.

Using °C and minutes as the units makes the most sense.

Separating the DE: $\int \frac{du}{u-20} = \int k\,dt$ (since $T_0 = 20$)

$$\ln|u-20| = kt + c$$

$u(0) = 95$ and $u(1) = 70$, so we can drop the absolute values:

$$u - 20 = e^{kt+c}$$

$$u = 20 + e^{kt+c}$$

$t=0$ gives: $95 = 20 + e^{0+c}$

$e^c = 75$

$c = \ln 75$ and $u = 20 + 75e^{kt}$

$t=1$ gives: $70 = 20 + 75e^{k\cdot 1}$

$e^k = \frac{50}{75} = \frac{2}{3}$

$\therefore k = \ln\left(\frac{2}{3}\right)$

and $u = 20 + 75\left(e^{\ln(\frac{2}{3})}\right)^t = 20 + 75\cdot\left(\frac{2}{3}\right)^t$

Letting $u = 45$: $45 = 20 + 75\cdot\left(\frac{2}{3}\right)^t$

$$\left(\frac{2}{3}\right)^t = \frac{25}{75} = \frac{1}{3}$$

$$t = \log_{\frac{2}{3}}\left(\frac{1}{3}\right) = \frac{\ln(1/3)}{\ln(2/3)} = \frac{-\ln 3}{\ln 2 - \ln 3}$$

9. Consider the series **(5 marks)**

$$\sum_{n=5}^{\infty} \frac{n!}{n^n}.$$

(a) **(2 marks)** Does this series converge or diverge?

Ratio Test: $\lim_{n\to\infty} \left(\frac{(n+1)!}{(n+1)^{n+1}}\right) \Big/ \left(\frac{n!}{n^n}\right) = \lim_{n\to\infty} \frac{\cancel{(n+1)!}^{\,n+1}}{\cancel{n!}} \cdot \frac{n^n}{(n+1)^{n+1}}$

$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{1+n}\right)^n = \frac{1}{\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1$

$\therefore$ Converges.

(b) **(3 marks)** Assuming that $\dfrac{n!}{n^n} < \dfrac{1}{2^n}$ for $n = 6, 7, \ldots$, find a value for $M > 0$ such that

$$\sum_{n=5}^{\infty} \frac{n!}{n^n} \leqslant M.$$

NOTE

$$\sum_{n=5}^{\infty} \frac{n!}{n^n} = \frac{5!}{5^5} + \sum_{n=6}^{\infty} \frac{n!}{n^n}$$

$$\leqslant \frac{5!}{5^5} + \sum_{n=6}^{\infty} \frac{1}{2^n}$$

$$= \frac{4!}{5^4} + \frac{1}{2^5} \qquad \left[= \frac{1{,}393}{20{,}000}\right]$$

10. Consider the series **(5 marks)**

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

(a) **(2 marks)** Does this series converge absolutely, conditionally, or diverge?

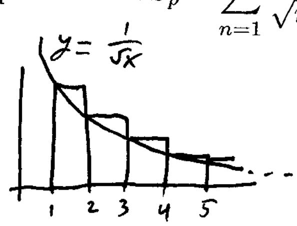
The series converges by the Alternating Series (né Leibniz) Test since $\frac{1}{\sqrt{n}}$ decreases to zero.

$\sum \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum \frac{1}{\sqrt{n}}$ diverges (p-series, $p = \frac{1}{2} \leq 1$).

$\therefore$ The series converges conditionally.

(b) **(3 marks)** Consider the partial sum $S_p = \sum_{n=1}^{p} \frac{1}{\sqrt{n}}$. For which value of p can you guarantee that $S_p \geqslant 2016$?

A reminder:



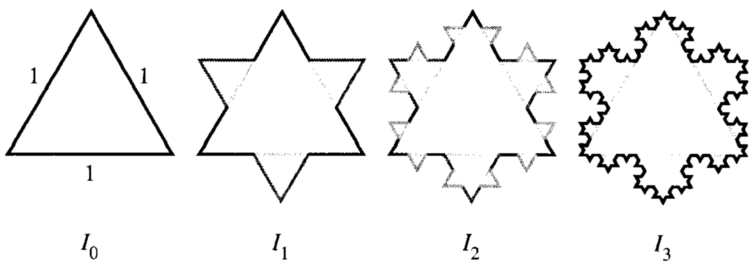
$$S_p = \sum_{n=1}^{p} \frac{1}{\sqrt{n}} \geq \int_1^{p+1} \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_1^{p+1} = 2\left(\sqrt{p+1} - 1\right)$$

Let $2(\sqrt{p+1} - 1) \geq 2016$

$\sqrt{p+1} \geq 1008 + 1 = 1009$

$p \geq 1009^2 - 1 \quad [= 1{,}018{,}080]$

11. Let us study the Snowflake Fractal. **(10 marks)**



To construct this fractal we:

- Start with an equilateral triangle I_0 of side length 1
- Remove the middle third of each side of I_0 and replace it with a new outward equilateral triangle with sides of length $\frac{1}{3}$ (see I_1 in the figure)
- Remove the middle third of each side of I_1 and replace it with a new outward equilateral triangle with sides of length $\frac{1}{3^2}$ (see I_2 in the figure)
- Repeat this process where I_{n+1} is obtained by replacing the middle third of each side of I_n by a new outward equilateral triangle with sides of length $\frac{1}{3^{n+1}}$

The limiting figure as $n \to \infty$ is called the Snowflake Fractal.

Let L_n be the perimeter of I_n. Note $L_0 = 3$.

(a) **(2 marks)** Calculate L_1.

Note that we remove $\frac{1}{3}$ of each side and replace it with two line segments equal to the original.

so, $L_1 = L_0 - \frac{1}{3}L_0 + 2 \cdot \frac{1}{3}L_0 = \frac{4}{3}L_0 = 4$

$L_1 = \boxed{4}$

(b) **(2 marks)** Calculate L_2.

By the same argument, $L_2 = \frac{4}{3} \cdot L_1 = \frac{16}{3}$

$L_2 = \frac{16}{3}$

(c) **(2 marks)** Calculate L_3.

Ditto.

$L_3 = \frac{64}{9}$

(d) (2 marks) Find a formula for L_n.

Ditto ^{n}.

$L_n = 3 \cdot \left(\frac{4}{3}\right)^n$

(e) (2 mark) Does L_n converge? If so, calculate its limit.

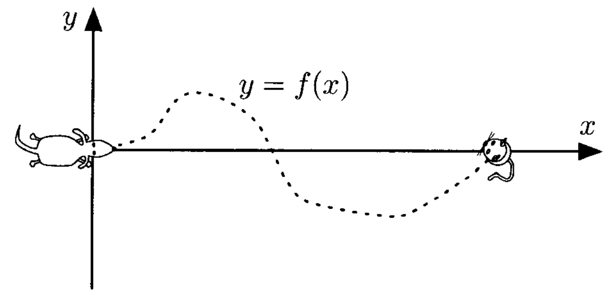
L_n is geometric, with $r > 1$, so it diverges.

12. A cat named Taylor is planning his escape from a dog called Maclaurin. **(10 marks)**

His plan is to wait until the dog almost reaches his position (2,0), and then jump away quickly to avoid the dog.

Before deciding where to go, Taylor (the cat) takes one good look at the dog and calculates a few derivatives of the function

$$f(x) = \text{path the dog intends to take.}$$



(the dotted path is not accurate; it's just an example)

(a) (3 marks) Assume that the cat calculates the following

$$f(0) = 0 \qquad f'(0) = \frac{1}{2}$$

$$f''(0) = -1 \qquad f'''(0) = \frac{1}{4}$$

Give an expression that approximates the path $f(x)$ of the dog using all this information.

$$f(x) \approx \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$= \frac{0}{1} + \frac{(1/2)}{1}x + \frac{-1}{2}\cdot x^2 + \frac{(1/4)}{6}x^3$$

$$= \frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{24}$$

(b) **(2 marks)** Using the calculations from **(a)**, help Taylor estimate the derivative $f'(2)$.

$$f'(x) \approx \frac{1}{2} - x + \frac{x^2}{8}$$

$$\therefore f'(2) \approx \frac{1}{2} - 2 + \frac{1}{2} = -1$$

(c) **(1 mark)** As the dog approaches the cat, from the cat's perspective is the dog moving to the right (towards positive y) or to the left (towards negative y)?

$f'(2) < 0$, so $f(x)$ is decreasing (moving toward negative y), $\therefore$ The dog is moving to the left.

(d) **(3 marks)** Taylor (the cat) knows from previous chases that Maclaurin's (the dog's) agility only allows the dog a $\boxed{\text{maximum } |f^{(4)}(x)| \text{ of } 3.}$

Estimate the maximum error in the cat's approximation of $f'(2)$.

(**Hint.** Observe that we want the error of the approximation of the derivative of f at $x = 2$)

Recall $f'(x) \approx \frac{1}{2} - x + \frac{x^2}{8}$

$\therefore \ |R_n(x)| \leq \frac{M}{3!}|x-0|^3$, where $\boxed{M = 3}$.

$$|R_n(2)| \leq \frac{3 \cdot 2^3}{3!}$$

$$= 4$$

(e) **(1 mark)** Recall that the cat's plan is to jump out of the way at the last instant.

Should the cat trust his approximation to make a decision? Justify using your answer from **(d)**

No. $|R_n(2)| > |f'(2)|$ estimate, so the actual value may be positive instead of negative. Also, everyone knows that cats are allergic to mathematics and have never advanced past basic fractions. Leibniz, on the other hand, was a Great Dane.