University of Toronto
Solutions to the **MAT187H1S TERM TEST**
of **Thursday, March 13, 2014**
Duration: 100 minutes

General Comments:

1. Many students have serious problems handling logs and exponentials. If you don't know the properties of logs and exponentials, you had better review and learn them! In particular, correct work with logs and exponentials is crucial to Questions 1, 2, 5, 6 & 8. Similarly there were lots of students who had difficulty handling the absolute value signs in Questions 2 and 8.

2. In Question 2 it is not actually necessary to solve for T in general.

3. In Questions 1 or 5, a constant of integration divided by the integrating factor is *not* a constant!

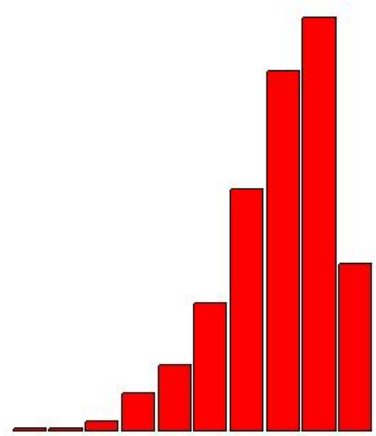
4. If $\lim_{k\to\infty} u_k \neq 0$ then the series $\sum u_k$ diverges. However, the converse of this statement is *not* true. Any 'solution' that says an infinite series converges because the kth term goes to zero is worthless.

5. No marks were awarded in Question 6 if you stated that the series diverges without giving any justification. You could get part marks if your justification is potentially correct but incompletely presented.

6. Many students wrote down very confusing 'explanations' for Questions 4, 6 & 7. There is no obligation on our part to read things that don't make sense; you certainly won't get marks for it!

Breakdown of Results: 460 students wrote this test. The marks ranged from 5% to 100%, and the average was 72.7%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	11.7%
A	40.6%	80-89%	28.9 %
B	25.2%	70-79%	25.2%
C	17.0%	60-69%	17.0%
D	8.9%	50-59%	8.9%
F	8.3%	40-49%	4.6%
		30-39%	2.6%
		20-29%	0.7%
		10-19%	0.2%
		0-9%	0.2%



1. [7 marks] Solve the initial value problem for y as a function of x :

$$\frac{dy}{dx} + \frac{3x\,y}{x^2+4} = x;\ \ y = 1 \text{ when } x = 0.$$

Solution: use the method of the integrating factor.

$$\mu = e^{\int 3x/(x^2+4)\,dx} = e^{3/2\ln(x^2+4)} = (x^2+4)^{3/2}.$$

Hence

$$\begin{aligned} y(x^2+4)^{3/2} &= \int x(x^2+4)^{3/2}\,dx = \frac{1}{2}\frac{(x^2+4)^{5/2}}{5/2} + C = \frac{1}{5}(x^2+4)^{5/2} + C \\ \Rightarrow y &= \frac{1}{5}(x^2+4) + \frac{C}{(x^2+4)^{3/2}} \end{aligned}$$

Use the initial condition to find C: let $x = 0, y = 1$. Then

$$1 = \frac{4}{5} + \frac{C}{4^{3/2}} \Rightarrow C = \frac{8}{5}.$$

Thus

$$y = \frac{1}{5}(x^2+4) + \frac{8}{5\,(x^2+4)^{3/2}}.$$

2. [8 marks] Newton's Law of Cooling states that

$$\frac{dT}{dt} = -k(T - A),$$

where T is the temperature of an object at time t, in a room with constant ambient temperature A, and k is a positive constant.

Suppose a cup of coffee, initially at temperature 95 C, is placed on a table in a room with constant temperature 20 C. If the temperature of the coffee is 75 C ten minutes later, when will the temperature of the coffee be 50 C?

Solution: use $A = 20$ and let t be in minutes. Separate variables:

$$\int \frac{dT}{T-20} = -\int k\, dt \Leftrightarrow \ln|T - 20| = -kt + C.$$

To find C use the initial conditions, $t = 0, T = 95$:

$$\ln 75 = C.$$

To find k use the fact that at $t = 10, T = 75$:

$$\ln 55 = -10k + \ln 75 \Rightarrow k = \frac{1}{10}\ln\left(\frac{15}{11}\right) \approx 0.0310$$

Now let $T = 50$ and solve for t:

$$\begin{aligned} & \ln 30 = -\frac{t}{10}\ln\left(\frac{15}{11}\right) + \ln 75 \\ \Rightarrow\quad & \frac{t}{10}\ln\left(\frac{15}{11}\right) = \ln 75 - \ln 30 = \ln\left(\frac{5}{2}\right) \\ \Rightarrow\quad & t = 10\frac{\ln(5/2)}{\ln(15/11)} \approx 29.543 \end{aligned}$$

So it will take about 29.543 minutes until the temperature of the coffee is 50 C.

3. [7 marks] Solve the initial value problem

$$x''(t) + 6\,x'(t) + 13\,x(t) = 0; \quad x(0) = 3; \quad x'(0) = -1$$

for x as a function of t.

Solution: the auxiliary quadratic is $r^2 + 6r + 13$. Solve:

$$r^2 + 6r + 13 = 0 \Leftrightarrow r = \frac{-6 \pm \sqrt{36 - 52}}{2} = -3 \pm 2i.$$

Thus

$$x = C_1 e^{-3t} \cos(2t) + C_2 e^{-3t} \sin(2t).$$

To find C_1 use the initial condition $x = 3$ when $t = 0$:

$$3 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 3.$$

To find C_2 you need to find x' :

$$x' = C_1(-3e^{-3t} \cos(2t) - 2e^{-3t} \sin(2t)) + C_2(-3e^{-3t} \sin(2t) + 2e^{-3t} \cos(2t)).$$

Now substitute $t = 0, x' = -1, C_1 = 3$:

$$-1 = 3(-3) + 2C_2 \Leftrightarrow C_2 = \frac{8}{2} = 4.$$

Thus

$$x = 3e^{-3t} \cos(2t) + 4e^{-3t} \sin(2t).$$

4. [8 marks] Decide if each of the following infinite series converges or diverges. Mark your choice to the right, and **justify** your choice in the space provided.

(a) [2 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ ○ Converges ⊗ Diverges

Justification: this is a p-series, with $p = 1/2 < 1$; therefore it diverges.

(b) [3 marks] $\displaystyle\sum_{k=2}^{\infty} \frac{\ln k}{4^k}$ ⊗ Converges ○ Diverges

Justification: use the ratio test.

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\ln(k+1)}{\ln k} \frac{4^k}{4^{k+1}} = \frac{1}{4} \lim_{k\to\infty} \frac{k}{k+1} = \frac{1}{4} \cdot 1 = \frac{1}{4} < 1.$$

(c) [3 marks] $\displaystyle\sum_{m=2}^{\infty} \frac{1}{m^3 \ln m}$ ⊗ Converges ○ Diverges

Justification: use the comparison test, with $a_m = \dfrac{1}{m^3 \ln m}$, $b_m = \dfrac{1}{m^3}$. Then

$$\sum_{m=2}^{\infty} b_m = \sum_{m=2}^{\infty} \frac{1}{m^3}$$

converges, because it is a p-series with $p = 3 > 1$, and

$$a_m < b_m$$

since

$$\frac{1}{m^3 \ln m} < \frac{1}{m^3}$$

for $m \geq 3$. Thus $\displaystyle\sum_{m=2}^{\infty} \frac{1}{m^3 \ln m}$ converges by the comparison test.

5. [11 marks] Recall: if $x(t)$ is the mass of solute dissolved in a solution of volume $V(t)$, at time t, in a large mixing tank, then

$$\frac{d\,x(t)}{dt} + \frac{r_o\,x(t)}{V(t)} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 500 L of pure water. At $t = 0$, a salt water solution containing 0.2 kg salt per L is added at a rate of 15 L/min and the mixed solution is drained off at a rate of 10 L/min. How much salt is in the tank when it reaches the point of overflowing?

Solution: We have $r_i = 15, r_0 = 10, c_i = 0.2 = 1/5$, so $V = 500 + (r_i - r_0)t = 500 + 5t$. Hence the differential equation is

$$\frac{dx}{dt} + \frac{10\,x}{500+5t} = \frac{15}{5} \Leftrightarrow \frac{dx}{dt} + \frac{2x}{100+t} = 3.$$

Use the method of the integrating factor, with

$$\mu = e^{\int \frac{2dt}{100+t}} = e^{2\ln(100+t)} = (100+t)^2.$$

Then

$$x = \frac{1}{(100+t)^2}\int 3(100+t)^2\,dt = \frac{1}{(100+t)^2}\left((100+t)^3 + C\right)$$

$$\Rightarrow x = 100 + t + \frac{C}{(100+t)^2}.$$

At $t = 0, x = 0$, so

$$0 = 100 + \frac{C}{100^2} \Rightarrow C = -100^3.$$

Finally, $V = 1000 \Leftrightarrow 500 + 5t = 1000 \Leftrightarrow t = 100$, and at that time:

$$x = 100 + 100 - \frac{100^3}{(100+100)^2} = 175,$$

exactly. So there will be 175 kg of salt in the tank when it reaches the point of overflowing.

6. [4 marks] Does the infinite series

$$\sum_{n=2}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

converge or diverge? You must justify your answer to get full marks!

Solution: the n-th term is

$$a_n = \ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln n;$$

so the series is a telescoping series. Its N-th partial sum is

$$S_N = a_2 + \cdots + a_N = \ln 3 - \ln 2 + \ln 4 - \ln 3 + \cdots + \ln(N+1) - \ln N = \ln(N+1) - \ln 2.$$

But then

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} (\ln(N+1) - \ln 2) = \infty$$

and the series diverges.

Alternate Solution: let $a_n = \ln\left(\frac{n+1}{n}\right) = \ln\left(1+\frac{1}{n}\right)$; $b_n = \frac{1}{n}$. Then use the limit comparison test:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\frac{1}{1+1/n}(-1/n^2)}{-1/n^2} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1.$$

Since $\sum b_n$ is the harmonic series, it diverges. By LCT, so does $\sum a_n$.

Alternate Solution: let $f(x) = \ln(1+1/x)$ and use the integral test: if $x \geq 2$ then $1 + 1/x > 1$ so $f(x) = \ln(1+1/x) > 0$; and

$$f'(x) = -\frac{1}{x(x+1)} < 0$$

so f is decreasing as well. Now use integration by parts with $u = f(x), dv = dx$:

$$\begin{aligned}
\int_2^\infty f(x)\,dx &= [xf(x)]_2^\infty + \int_2^\infty \frac{dx}{x+1} \\
&= \lim_{x\to\infty} x\ln(1+1/x) - 2\ln 5.5 + \lim_{x\to\infty} \ln(x+1) - \ln 3 \\
&= 1 - 2\ln 5.5 - \ln 3 + \infty \\
&= \infty
\end{aligned}$$

so the original series diverges by the integral test.

7. [8 marks; 4 marks for each part.] Determine if each of the following infinite series converges conditionally, converges absolutely, or diverges.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{2^n}$

Solution: converges absolutely, by the ratio test:

$$\rho = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^3}{2^{n+1}}\frac{2^n}{n^3} = \frac{1}{2}\lim_{n\to\infty}\frac{(n+1)^3}{n^3} = \frac{1}{2}\cdot 1 = \frac{1}{2} < 1.$$

(b) $\sum_{k=1}^{\infty}(-1)^k \frac{k}{k^2+3k+5}$

Solution: converges conditionally. The series

$$\sum_{k=1}^{\infty}(-1)^k \frac{k}{k^2+3k+5}$$

converges by the alternating series test, since

$$\lim_{k\to\infty}\frac{k}{k^2+3k+5} = \lim_{k\to\infty}\frac{1/k}{1+3/k+5/k^2} = 0$$

and

$$\begin{aligned} a_k > a_{k+1} \quad &\Leftrightarrow \quad \frac{k}{k^2+3k+5} > \frac{k+1}{(k+1)^2+3(k+1)+5} \\ &\Leftrightarrow \quad k^3+5k^2+9k > k^3+4k^2+8k+5 \\ &\Leftrightarrow \quad k^2+k > 5 \end{aligned}$$

and this last statement is true if $k \geq 2$. But the series

$$\sum_{k=1}^{\infty}\left|(-1)^k \frac{k}{k^2+3k+5}\right| = \sum_{k=1}^{\infty}\frac{k}{k^2+3k+5}$$

diverges by the limit comparison test, using

$$a_k = \frac{k}{k^2+3k+5}, \quad b_k = \frac{1}{k}.$$

That is, $\sum b_k$ is the harmonic series, which diverges, and

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{k^2}{k^2+3k+5} = 1.$$

8. [7 marks.] Let M be the mass of a bacterial colony at time t, with $M = M_0$ when $t = 0$. Suppose the growth of the colony is described by the Gompertz growth equation,

$$\frac{dM}{dt} = -rM \ln\left(\frac{M}{K}\right),$$

for some positive constants r and K, with $0 < M_0 < K$. Solve the above initial value problem given that $r = 1, K = 4, M_0 = 1$. Graph your solution.

Solution: separate variables.

$$\int \frac{dM}{M(\ln M - \ln 4)} = -\int dt \Leftrightarrow \ln|\ln M - \ln 4| = -t + C.$$

To find C let $t = 0, M_0 = 1$:

$$\ln|\ln 1 - \ln 4| = C \Leftrightarrow C = \ln(\ln 4).$$

Thus

$$|\ln M - \ln 4| = e^{-t+\ln(\ln 4)} = e^{-t}\ln 4 \Rightarrow \ln M - \ln 4 = \pm e^{-t}\ln 4.$$

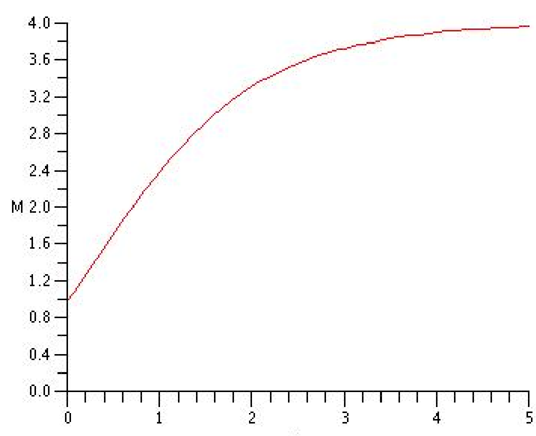
But to match the initial condition $M = 1$ at $t = 0$ you have to pick the negative option; that is:

$$\ln M = \ln 4 - e^{-t}\ln 4$$

and then

$$M = e^{(1-e^{-t})\ln 4} = 4^{1-e^{-t}}.$$

Graph:



The graph has a horizontal asymptote at $M = 4$ and an inflection point at $M = 4/e$.