University of Toronto
Solutions to the **MAT187H1S TERM TEST**
of **Thursday, March 14, 2013**
Duration: 100 minutes

General Comments:

1. In Question 1, if you don't get the correct integrating factor you may well have to integrate something that *can't* be integrated! This would basically result in a forfeit of 7 marks.

2. In Question 2 if you don't handle the absolute value signs properly—and a lot of students didn't—you will get nonsense, such as $\ln(-1)$ which is not defined, or $y = 4/(1 - e^{4x^2-4})$, which is not defined at $x = 1$.

3. Many students could not solve the quadratic in Question 3 properly!

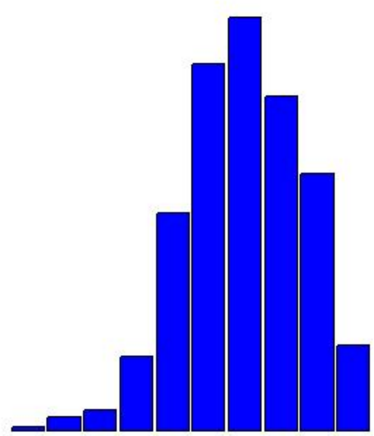
4. If $\lim_{k\to\infty} u_k \neq 0$, then the infinite series $\sum u_k$ diverges. However, the converse of this statement is *not* true. Any solution that says an infinite series converges because the kth term goes to zero is worthless. Similarly, if $a_k < 1/k$, then the fact that the harmonic series diverges tells you *nothing* about $\sum a_k$; any conclusion about $\sum a_k$ is worthless.

5. Many students had real difficulty using proper notation in Question 7, or explaining what they were doing. Many students equated a series to the limit of its kth term, or a fraction to its limit. Some failed to use the ratio test properly, or did not prove the *two* hypotheses of the alternating series test. You must write and explain well!

6. Question 8 was right out of the homework.

Breakdown of Results: 502 students wrote this test. The marks ranged from 8.3% to 98.3%, and the average was 63.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.8%
A	19.1%	80-89%	14.3%
B	18.7%	70-79%	18.7%
C	23.1%	60-69%	23.1%
D	20.5%	50-59%	20.5%
F	18.6%	40-49%	12.2%
		30-39%	4.2%
		20-29%	1.2%
		10-19%	0.8%
		0-9%	0.2%



1. [8 marks] Solve the initial value problem for y as a function of x :

$$\frac{dy}{dx} + y\ \tan x = \sec x + \cos x;\ \ y = 7 \text{ when } x = \pi.$$

Solution: use the method of the integrating factor.

$$\mu = e^{\int \tan x\, dx} = e^{\ln|\sec x|} = |\sec x|.$$

Multiply the original equation by either $\sec x$ or $-\sec x$, it will make no difference:

$$\sec x \frac{dy}{dx} + y\ \sec x \tan x = \sec^2 x + \sec x \cos x \Leftrightarrow \frac{d(y \sec x)}{dx} = \sec^2 x + 1.$$

Hence

$$\begin{aligned} y \sec x &= \int (\sec^2 x + 1)\, dx = \tan x + x + C \\ \Rightarrow y &= \cos x(\tan x + x + C) \\ \Rightarrow y &= \sin x + x\ \cos x + C\ \cos x \end{aligned}$$

Use the initial condition to find C:

$$7 = \sin \pi + \pi \cos \pi + C \cos \pi \Leftrightarrow 7 = -\pi - C \Leftrightarrow C = -7 - \pi.$$

Thus

$$y = \sin x + x\ \cos x - (7 + \pi) \cos x.$$

2. [8 marks] Solve the initial value problem for y as a function of x :

$$\frac{dy}{dx} = 2x(y^2 - 4y); \; y = 2 \text{ when } x = 1.$$

Solution: separate variables:

$$\int \frac{dy}{y^2 - 4y} = \int 2x\, dx \Leftrightarrow \int \frac{dy}{y(y-4)} = x^2 + C.$$

Then integral on the left can be done by partial fractions:

$$\int \frac{dy}{y(y-4)} = \int \left(\frac{1/4}{y-4} - \frac{1/4}{y} \right) dy = \frac{1}{4} \ln \left| \frac{y-4}{y} \right|.$$

To find C use the initial conditions:

$$\frac{1}{4} \ln \left| \frac{2-4}{2} \right| = 1 + C \Leftrightarrow 0 = 1 + C \Leftrightarrow C = -1.$$

Therefore:

$$\frac{1}{4} \ln \left| \frac{y-4}{y} \right| = x^2 - 1.$$

But you are not finished as you must still solve for y :

$$\begin{aligned}
\frac{1}{4} \ln \left| \frac{y-4}{y} \right| = x^2 - 1 \quad &\Rightarrow \quad \ln \left| \frac{y-4}{y} \right| = 4x^2 - 4 \\
&\Rightarrow \quad \left| \frac{y-4}{y} \right| = e^{4x^2-4} \\
\text{(Note: at initial condition, } (y-4)/y < 0) \quad &\Rightarrow \quad \frac{4-y}{y} = e^{4x^2-4} \\
&\Rightarrow \quad \frac{4}{y} = 1 + e^{4x^2-4} \\
&\Rightarrow \quad y = \frac{4}{1 + e^{4x^2-4}}
\end{aligned}$$

So

$$y = \frac{4}{1 + e^{4x^2-4}}.$$

3. [8 marks] Solve the initial value problem

$$x''(t) + 2\,x'(t) + 10\,x(t) = 0; \quad x(0) = 2; \quad x'(0) = 1$$

for x as a function of t.

Solution: the auxiliary quadratic is $r^2 + 2r + 10$. Solve:

$$r^2 + 2r + 10 = 0 \Leftrightarrow r = \frac{-2 \pm \sqrt{4 - 40}}{2} = -1 \pm 3i.$$

Thus

$$x = C_1 e^{-t} \cos(3t) + C_2 e^{-t} \sin(3t).$$

To find C_1 use the initial condition $x = 2$ when $t = 0$:

$$2 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 2.$$

To find C_2 you need to find x' :

$$x' = C_1(-e^{-t}\cos(3t) - 3e^{-t}\sin(3t)) + C_2(-e^{-t}\sin(3t) + 3e^{-t}\cos(3t)).$$

Now substitute $t = 0, x' = 1, C_1 = 2$:

$$1 = 2(-1 - 0) + C_2(0 + 3) \Leftrightarrow 3C_2 = 3 \Leftrightarrow C_2 = 1.$$

Thus

$$x = 2e^{-t}\cos(3t) + e^{-t}\sin(3t).$$

4. [8 marks] Decide if each of the following infinite series converges or diverges. Mark your choice to the right, and **justify** your choice in the space provided.

(a) [2 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^4}$ ⊗ Converges ○ Diverges

Justification: this is a p-series, with $p = 4 > 1$; therefore it converges.

(b) [3 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{k^2}{3k^4 + 7k^2 + 17}$ ⊗ Converges ○ Diverges

Justification: use the limit comparison test with $b_k = \dfrac{1}{3k^2}$:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{3k^4}{3k^4 + 7k^2 + 17} = 1.$$

Since $\sum b_k$ converges (p-series with $p = 2 > 1$) so does $\sum a_k$. You could also use the comparison test, since

$$a_k < b_k.$$

(c) [3 marks] $\displaystyle\sum_{m=2}^{\infty} \frac{1}{m \ln m}$ ○ Converges ⊗ Diverges

Justification: use the integral test, and let $u = \ln x$.

$$\int_2^{\infty} \frac{dx}{x \ln x} = \int_{\ln 2}^{\infty} \frac{du}{u} = \lim_{b\to\infty} [\ln u]_{\ln(2)}^{b} = \infty$$

So the series diverges by the integral test.

5. [10 marks] Recall: if $x(t)$ is the mass of solute dissolved in a solution of volume $V(t)$, at time t, in a large mixing tank, then

$$\frac{d\,x(t)}{dt} + \frac{r_o\,x(t)}{V(t)} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 500 L of water and 100 kg of dissolved salt. At $t = 0$, a salt water solution containing 0.1 kg salt per L is added at a rate of 20 L/min and the mixed solution is drained off at a rate of 10 L/min. How much salt is in the tank when it reaches the point of overflowing?

Solution: We have $r_i = 20, r_0 = 10, c_i = 1/10$, so $V = 500 + (r_i - r_0)t = 500 + 10t$. Hence the differential equation is

$$\frac{dx}{dt} + \frac{10\,x}{500+10t} = \frac{20}{10} \Leftrightarrow \frac{dx}{dt} + \frac{x}{50+t} = 2.$$

Use the method of the integrating factor, with

$$\mu = e^{\int \frac{dt}{50+t}} = e^{\ln(50+t)} = 50 + t.$$

Then

$$x = \frac{1}{50+t}\int (50+t)2\,dt = \frac{1}{50+t}\left(100t + t^2 + C\right)$$

$$\Rightarrow x = \frac{100t + t^2 + C}{50+t}.$$

At $t = 0, x = 100$, so

$$100 = \frac{C}{50} \Rightarrow C = 5000.$$

Finally, $V = 1000 \Leftrightarrow 500 + 10t = 1000 \Leftrightarrow t = 50$, and at that time:

$$x = \frac{100(50) + 50^2 + 5000}{100} = 125,$$

exactly. So there will be 125 kg of salt in the tank when it reaches the point of overflowing.

6. [4 marks] A common mistake of beginning calculus students is to write

$$\int \frac{dx}{f(x)} = \ln|f(x)| + C.$$

Find **all** differentiable functions $f(x)$ for which this formula is actually true.

Solution: Suppose the formula is true. Differentiate both sides:

$$\begin{aligned}\frac{1}{f(x)} = \frac{f'(x)}{f(x)} \quad &\Rightarrow \quad f'(x) = 1 \\ &\Rightarrow \quad f(x) = \int 1\,dx \\ &\Rightarrow \quad f(x) = x + C\end{aligned}$$

7. [8 marks; 4 marks for each part.] Determine if each of the following infinite series converges conditionally, converges absolutely, or diverges.

(a) $\sum_{n=1}^{\infty}(-1)^n \frac{n!}{n^n}$

Solution: converges absolutely, by the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!}\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty}\frac{(n+1)}{(n+1)}\left(\frac{n}{n+1}\right)^n$$

$$= \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-n} = \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^{-1} = \frac{1}{e} < 1.$$

(b) $\sum_{k=1}^{\infty}(-1)^k \frac{1}{k+\sin^2 k}$

Solution: converges conditionally. The series

$$\sum_{k=1}^{\infty}(-1)^k \frac{1}{k+\sin^2 k}$$

converges by the alternating series test, since

$$\lim_{k\to\infty}\frac{1}{k+\sin^2 k} = 0$$

and

$$a_k > a_{k+1} \Leftrightarrow k+1+\sin^2(k+1) > k+\sin^2 k \Leftrightarrow 1 > \sin^2 k - \sin^2(k+1),$$

and this last statement is true because the difference of any two positive distinct numbers between 0 and 1 is less than 1.

But the series

$$\sum_{k=1}^{\infty}\left|(-1)^k \frac{1}{k+\sin^2 k}\right| = \sum_{k=1}^{\infty}\frac{1}{k+\sin^2 k}$$

diverges by the limit comparison test, using

$$a_k = \frac{1}{k+\sin^2 k}, \quad b_k = \frac{1}{k}.$$

That is, $\sum b_k$ is the harmonic series, which diverges, and

$$\lim_{k\to\infty}\frac{a_k}{b_k} = \lim_{k\to\infty}\frac{k}{k+\sin^2 k} = \lim_{k\to\infty}\frac{1}{1+(\sin^2 k)/k} = \frac{1}{1+0} = 1.$$

8. [6 marks.] A ball is dropped from a height of 10 m. Each time it strikes the ground it bounces up vertically to a height that is $\frac{3}{4}$ of the preceding height. Find the total distance the ball will travel if it is assumed to bounce infinitely often.

Solution: the total distance travelled by the ball is

$$
\begin{aligned}
D &= 10 + 2\left(\frac{3}{4}\right)10 + 2\left(\frac{3}{4}\right)^2 10 + 2\left(\frac{3}{4}\right)^3 10 + \cdots \\
&= 10 + 2\left(\frac{3}{4}\right)10\left(1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots\right) \\
&= 10 + 15\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n \\
&= 10 + 15\left(\frac{1}{1-3/4}\right) \\
&= 10 + 15(4) \\
&= 70
\end{aligned}
$$