University of Toronto
Solutions to the **MAT187H1S TERM TEST**
of **Thursday, March 15, 2012**
Duration: 100 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

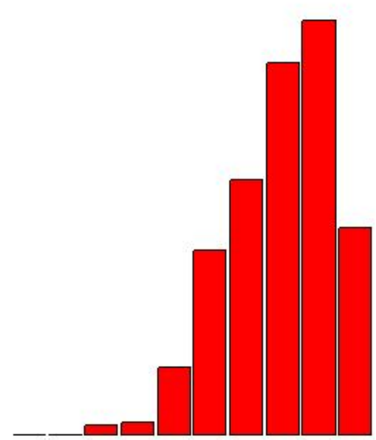
Instructions: Make sure this test contains 8 pages. Do not remove any pages from this test. Answer all questions. Present your solutions in the space provided. This means: show your work! The value for each question is indicated in parentheses beside the question number. **TOTAL MARKS: 60**

General Comments:

1. In Question 1 you must be careful with the constants of integration. For example, the solution to part (a) is $y = e^{-x} \ln|x-1| + Ce^{-x}$, not $y = e^{-x} \ln|x-1| + C$.
2. If $\lim_{k\to\infty} u_k \neq 0$, then the infinite series $\sum u_k$ diverges. However, the converse of this statement is *not* true. Any solution that says an infinite series converges because the kth term goes to zero is worthless.
3. For the telescoping series in Question 7(b) you must calculate the Nth partial sum to get full marks. Canceling the first few terms in an infinite series is never enough to establish its convergence. Think of $\sum(-1)^n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$, which diverges.
4. The most common cause of lost marks on this test was high school algebra! There are still students who can't apply the quadratic formula correctly, or who think $1 + y^2 = (1+y)(1-y)$. We expect you to understand the material conceptually *and* to be able to calculate correctly.

Breakdown of Results: 471 students wrote this test. The marks ranged from 21.7% to 100%, and the average was 73.7%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	13.6%
A	40.8%	80-89%	27.2%
B	24.4%	70-79%	24.4%
C	16.8%	60-69%	16.8%
D	12.1%	50-59%	12.1%
F	5.9%	40-49%	4.5%
		30-39%	0.8%
		20-29%	0.6%
		10-19%	0.0%
		0-9%	0.0%



1. [9 marks] Find the general solution to each of the following differential equations:

(a) [5 marks] $\dfrac{dy}{dx} + y = \dfrac{1}{e^x - 1}$

Solution: use the method of the integrating factor.

$$\mu = e^{\int dx} = e^x,$$

$$e^x\, y = \int \frac{e^x\, dx}{e^x - 1} = \ln|e^x - 1| + C \Rightarrow y = e^{-x}\ln|e^x - 1| + Ce^{-x}.$$

(b) [4 marks] $\dfrac{dy}{dx} = \dfrac{1 + y^2}{x^2}$

Solution: separate variables.

$$\int \frac{dy}{1 + y^2} = \int \frac{dx}{x^2} \Rightarrow \tan^{-1} y = -\frac{1}{x} + C \Rightarrow y = \tan\left(-\frac{1}{x} + C\right).$$

2. [8 marks] Solve the initial value problem

$$x''(t) + 4\,x'(t) + 13\,x(t) = 0; \quad x(0) = -2; \quad x'(0) = 1$$

for x as a function of t.

Solution: the auxiliary quadratic is $r^2 + 4r + 13$. Solve:

$$r^2 + 4r + 13 = 0 \Leftrightarrow r = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i.$$

Thus

$$x = C_1 e^{-2t}\cos(3t) + C_2 e^{-2t}\sin(3t).$$

To find C_1 use the initial condition $x = -2$ when $t = 0:$

$$-2 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = -2.$$

To find C_2 you need to find $x':$

$$x' = C_1(-2e^{-2t}\cos(3t) - 3e^{-2t}\sin(3t)) + C_2(-2e^{-2t}\sin(3t) + 3e^{-2t}\cos(3t)).$$

Now substitute $t = 0, x' = 1, C_1 = -2:$

$$1 = -2(-2 - 0) + C_2(0 + 3) \Leftrightarrow 3C_2 = -3 \Leftrightarrow C_2 = -1.$$

Thus

$$x = -2e^{-2t}\cos(3t) - e^{-2t}\sin(3t).$$

3. [8 marks] Newton's Law of Cooling states that

$$\frac{dT}{dt} = k(T - A),$$

where T is the temperature of an object at time t, A is the constant temperature of the surrounding air, and k is a constant.

Suppose that at 11 AM a freshly baked pie is taken from the oven at temperature 200 C and placed onto a table in a kitchen with constant room temperature 20 C. Five min later the temperature of the pie is 100 C. When will the temperature of the pie be 25 C?

Solution: separate variables. Take $A = 20$; let $t = 0$ be 11 AM.

$$\int \frac{dT}{T - 20} = \int k dt \Rightarrow \ln |T - 20| = kt + C.$$

To find C let $t = 0, T = 200$:

$$\ln |200 - 20| = C \Leftrightarrow C = \ln 180.$$

To find k let $t = 5, T = 100$ and $C = \ln 180$:

$$\ln |100 - 20| = 5k + \ln 180 \Rightarrow 5k = \ln(4/9) \Rightarrow k = \frac{1}{5} \ln \left(\frac{4}{9}\right) \simeq -0.162186\ldots.$$

Finally, let $T = 25$ and solve for t:

$$\ln |25 - 20| = \frac{t}{5} \ln \left(\frac{4}{9}\right) + \ln 180 \Rightarrow t = -\frac{5 \ln 36}{\ln(4/9)} \simeq 22.1$$

So the pie will be 25 C at approximately 11:22 AM.

4. [10 marks] Recall: if $x(t)$ is the mass of solute dissolved in a solution of volume $V(t)$, at time t, in a large mixing tank, then

$$\frac{d\,x(t)}{dt} + \frac{r_o\,x(t)}{V(t)} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A tank with capacity 1000 L initially contains 400 L of water and 80 kg of dissolved salt. At $t = 0$, a salt water solution containing 0.1 kg salt per L is added at a rate of 15 L/min and the mixed solution is drained off at a rate of 10 L/min.

(a) [4 marks] What are the given values of r_o, r_i, c_i and $V(t)$?

$r_o = \underline{10}$ L/min, $r_i = \underline{15}$ L/min, $c_i = \underline{0.1}$ kg/L, $V(t) = \underline{400 + 5t}$ L

(b) [6 marks] How much salt is in the tank when it reaches the point of overflowing?

Solution: the differential equation is

$$\frac{dx}{dt} + \frac{10\,x}{400+5t} = \frac{15}{10} = \frac{3}{2}$$

which has integrating factor

$$\mu = e^{\int \frac{10dt}{400+5t}} = e^{2\ln(400+5t)} = (400+5t)^2.$$

Then

$$x = \frac{1}{(400+5t)^2}\int \frac{3}{2}(400+5t)^2\,dt = \frac{1}{(400+5t)^2}\left(\frac{3}{2}\frac{(400+5t)^3}{3}\cdot\frac{1}{5} + C\right)$$

$$\Rightarrow x = \frac{400+5t}{10} + \frac{C}{(400+5t)^2}.$$

At $t = 0, x = 80$, so

$$80 = \frac{400}{10} + \frac{C}{400^2} \Rightarrow C = 40(400^2).$$

Finally, $V = 1000 \Leftrightarrow 400 + 5t = 100 \Leftrightarrow t = 120$, and then

$$x = \frac{400+600}{10} + \frac{40(400^2)}{(400+600)^2} = 106.4,$$

exactly. So there will be 106.4 kg of salt in the tank when it reaches the point of overflowing.

5. [9 marks] Decide if each of the following infinite series converges or diverges. Mark your choice to the right, and justify your choice in the space provided.

(a) [3 marks] $\sum_{k=1}^{\infty} \frac{k^2}{3k^4 - 15k + 17}$ ⊗ Converges ○ Diverges

Justification: limit comparison test with $b_k = k^{-2}$:

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \lim_{k\to\infty} \frac{k^4}{3k^4 - 15k + 17} = \frac{1}{3}.$$

Since $\sum b_k$ converges (p-series with $p = 2 > 1$) so does $\sum a_k$.

(b) [3 marks] $\sum_{m=2}^{\infty} \frac{\ln m}{m^2}$ ⊗ Converges ○ Diverges

Justification: by comparison with the p-series, with $p = 3/2$. That is, use $\ln m < \sqrt{m}$:

$$a_m = \frac{\ln m}{m^2} < \frac{\sqrt{m}}{m^2} = \frac{1}{m^{3/2}} = b_m.$$

Or, the long way, use the integral test, with $u = \ln x$:

$$\int_2^{\infty} \frac{\ln x \, dx}{x^2} = \int_{\ln 2}^{\infty} e^{-u}\, u \, du = \lim_{b\to\infty} [-ue^{-u}]_{\ln 2}^{b} + \int_{\ln 2}^{\infty} e^{-u} du < \infty.$$

(c) [3 marks] $\sum_{n=2}^{\infty} \frac{n^{25}}{3^n}$ ⊗ Converges ○ Diverges

Justification: by the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{25}}{n^{25}} \frac{3^n}{3^{n+1}} = \frac{1}{3}\left(\lim_{n\to\infty} \frac{n+1}{n}\right)^{25} = \frac{1}{3} < 1$$

6. [8 marks; 4 marks for each part] Determine if the following infinite series converge conditionally, converge absolutely, or diverge.

(a) $\sum_{k=1}^{\infty}(-1)^k \frac{k+1}{k(k+3)}$

Solution: converges conditionally. The series

$$\sum_{k=1}^{\infty}(-1)^k \frac{k+1}{k(k+3)}$$

converges by the alternating series test, but the series

$$\sum_{k=1}^{\infty}\left|(-1)^k \frac{k+1}{k(k+3)}\right| = \sum_{k=1}^{\infty}\frac{k+1}{k(k+3)}$$

diverges by the limit comparison test, using

$$a_k = \frac{k+1}{k(k+3)}, \quad b_k = \frac{1}{k}.$$

That is, $\sum b_k$ is the harmonic series, which diverges, and

$$\lim_{k\to\infty}\frac{a_k}{b_k} = 1.$$

(b) $\sum_{k=1}^{\infty}(-1)^k \frac{(\ln k)^k}{e^{k^2}}$

Solution: converges absolutely, by the ratio test.

$$\lim_{k\to\infty}\sqrt[k]{\left|\frac{(-1)^k(\ln k)^k}{e^{k^2}}\right|} = \lim_{k\to\infty}\frac{\ln k}{e^k} = 0 < 1.$$

7. [8 marks; 4 for each part.] Find the exact sum of each of the following infinite series:

(a) $\displaystyle\sum_{k=0}^{\infty}\left(\frac{3}{2^k}-\frac{5}{3^{k+1}}\right)$

Solution: use the sum of an infinite geometric series, $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, if $|r|<1$.

$$\begin{aligned}
\sum_{k=0}^{\infty}\left(\frac{3}{2^k}-\frac{5}{3^{k+1}}\right) &= 3\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k - \frac{5}{3}\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k \\
&= \frac{3}{1-1/2} - \frac{5}{3}\cdot\frac{1}{1-1/3} \\
&= 6-\frac{5}{2} \\
&= \frac{7}{2} \text{ or } 3.5
\end{aligned}$$

(b) $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-1}$

Solution: this is a telescoping series. Use

$$a_n = \frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1}-\frac{1}{n+1}\right).$$

Then

$$\begin{aligned}
S_N &= a_1+a_2+a_3+\cdots+a_N \\
&= \frac{1}{2}\left(\frac{1}{1}-\frac{1}{3}+\frac{1}{2}-\frac{1}{4}+\frac{1}{3}-\frac{1}{5}+\frac{1}{4}-\frac{1}{6}+\cdots+\frac{1}{N-1}-\frac{1}{N+1}\right) \\
&= \frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{N}-\frac{1}{N+1}\right) \\
\Rightarrow \lim_{N\to\infty} S_N &= \lim_{N\to\infty}\frac{1}{2}\left(1+\frac{1}{2}-\frac{1}{N}-\frac{1}{N+1}\right) \\
&= \frac{1}{2}\left(\frac{3}{2}\right) \\
&= \frac{3}{4} \text{ or } 0.75
\end{aligned}$$