University of Toronto

Solutions to the **MAT187H1S TERM TEST**

of **Thursday, March 10, 2011**

Duration: 100 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Make sure this test contains 8 pages. Do not remove any pages from this test. Answer all questions. Present your solutions in the space provided. This means: show your work! The value for each question is indicated in parentheses beside the question number. **TOTAL MARKS: 60**

General Comments:

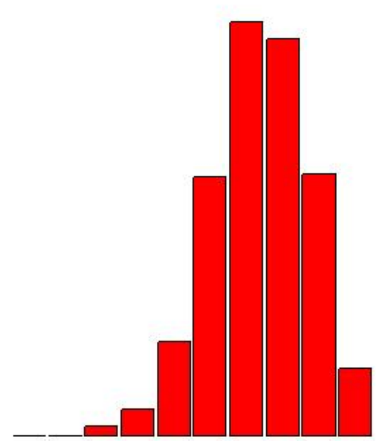
1. Many students are still confused about absolute values. In Question 1, the two equations
$$e^y = \frac{x^2}{2} - 7 \text{ and } y = \ln\left|\frac{x^2}{2} - 7\right|$$
are not the same. For example, $(0, \ln 7)$ is a solution to the equation on the right, but $(0, \ln 7)$ is *not* a solution to the equation on the left. The fact that the TA did not deduct a mark for this is not doing anybody any good! You can rest assured that confusion with absolute values on the final exam will cost you marks.

2. If $\lim_{k\to\infty} u_k \neq 0$, then the infinite series $\sum_{k=0}^{\infty} u_k$ diverges. However, the converse of this statement is *not* true. Any solution that says an infinite series converges because the kth term goes to zero is worthless.

3. In Question 7(b) many students used $\sqrt[n]{3^{-n} + 4^{-n}} = 3^{-1} + 4^{-1}$, or something equally untrue. Such algebraic bloopers will get you zero every time.

Breakdown of Results: 473 students wrote this test. The marks ranged from 23.3% to 98.3%, and the average was 67.6%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	4.4%
A	21.5%	80-89%	17.1%
B	26.0%	70-79%	26.0%
C	27.1%	60-69%	27.1%
D	16.9%	50-59%	16.9%
F	8.4%	40-49%	6.1%
		30-39%	1.7%
		20-29%	0.6%
		10-19%	0.0%
		0-9%	0.0%



1. [6 marks] Find the equation of a curve with x-intercept 4 whose tangent line at any point (x, y) has slope $x\,e^{-y}$.

Solution: Stated as an initial value problem, the question is: solve for y if

$$\frac{dy}{dx} = \frac{x}{e^y} \text{ and } y = 0 \text{ if } x = 4.$$

Separate variables;

$$\begin{aligned} \frac{dy}{dx} = \frac{x}{e^y} \quad &\Rightarrow \quad \int e^y\,dy = \int x\,dx \\ &\Rightarrow \quad e^y = \frac{x^2}{2} + C \\ &\Rightarrow \quad y = \ln\left(\frac{x^2}{2} + C\right) \end{aligned}$$

To find C, substitute $x = 4$ and $y = 0$:

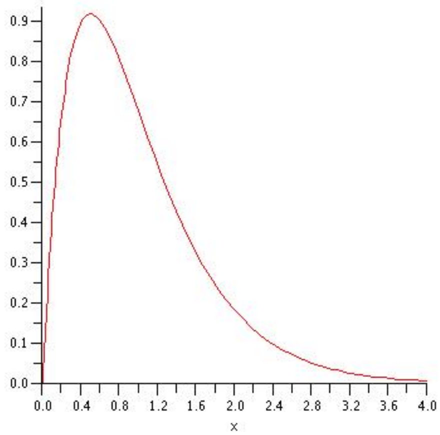
$$e^0 = \frac{16}{2} + C \Leftrightarrow C = -7.$$

Thus either of the following equations will do:

$$e^y = \frac{x^2}{2} - 7 \text{ or } y = \ln\left(\frac{x^2}{2} - 7\right).$$

2. [9 marks] The parts of this question are unrelated.

(a) [5 marks] Indicate whether or not each of the following differential equations could have a solution (y in terms of x) with a graph that looks like the graph below. It is not necessary to justify your choice.



(1) $y'' - 4y = 0$	⊗ No	○ Yes
(2) $y'' + 4y' + 4y = 0$	○ No	⊗ Yes
(3) $y'' - 3y' + 2y = 0$	⊗ No	○ Yes
(4) $y'' + 3y' + 2y = 0$	○ No	⊗ Yes
(5) $y'' + 4y = 0$	⊗ No	○ Yes

Solution: y must satisfy: $y(0) = 0; x > 0 \Rightarrow y > 0; y'(0) > 0$; and $\lim_{x\to\infty} y = 0$. The solutions for (1), $y = Ae^{2x} + Be^{-2x}$, for (3), $y = Ae^{x} + Be^{2x}$, and for (5), $y = A\cos(2x) + B\sin(2x)$, don't satisfy $\lim_{x\to\infty} y = 0$, unless $A = B = 0$.

The solutions for (2), $y = Ae^{-2x} + Bxe^{-2x}$, and for (4), $y = Ae^{-2x} + Be^{-x}$, can satisfy these conditions. For (2), take $A = 0$ and $B > 0$. For (4), take $B = -A$ and $A < 0$.

(b) [4 marks] Consider the initial value problem

$$\frac{dy}{dx} = 2xe^{x^2}(1 + y^2); \quad y(0) = \sqrt{3}.$$

Find the only critical point of y and determine if it is a maximum or a minimum point.

Solution:

$$\frac{dy}{dx} = 2x\, e^{x^2}(1 + y^2) \Rightarrow \begin{cases} \dfrac{dy}{dx} > 0 & \Leftrightarrow \quad x > 0 \\ \dfrac{dy}{dx} = 0 & \Leftrightarrow \quad x = 0 \\ \dfrac{dy}{dx} < 0 & \Leftrightarrow \quad x < 0 \end{cases}$$

Thus the only critical point, $(0, \sqrt{3})$, is a minimum point, by the first derivative test.

3. [8 marks] Solve the initial value problem

$$x\frac{dy}{dx} - y = 4x^2; \quad y(3) = -12$$

for y as a function of x.

Solution: use the method of the integrating factor. But first you have to rewrite the given differential equation as

$$\frac{dy}{dx} - \frac{y}{x} = 4x,$$

from which the integrating factor is

$$\mu = e^{\int (-1/x)\,dx} = e^{-\ln|x|} = \frac{1}{|x|}.$$

Take either $\mu = \pm 1/x$; then the general solution is

$$\begin{aligned} y &= \frac{\int \mu\, 4x\, dx}{\mu} \\ &= x \int 4\, dx \\ &= x\,(4x + C) \\ &= 4x^2 + Cx \end{aligned}$$

Particular Solution: let $x = 3, y = -12$ and solve for C:

$$-12 = 36 + 3C \Leftrightarrow C = -16.$$

Thus:

$$y = 4x^2 - 16x.$$

4. [9 marks] Solve for y as a function of x if $y'' + 4y' + 13y = 0$ and $y(0) = 2, y'(0) = 5$.

Solution: the auxiliary quadratic is $r^2 + 4r + 13$. Solve:

$$r^2 + 4r + 13 = 0 \Leftrightarrow r = \frac{-4 \pm \sqrt{16 - 52}}{2} = -2 \pm 3i.$$

Thus

$$y = C_1 e^{-2x} \cos(3x) + C_2 e^{-2x} \sin(3x).$$

To find C_1 use the initial condition $y = 2$ when $x = 0$:

$$2 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 2.$$

To find C_2 you need to find y' :

$$y' = C_1(-2e^{-2x} \cos(3x) - 3e^{-2x} \sin(3x)) + C_2(-2e^{-2x} \sin(3x) + 3e^{-2x} \cos(3x)).$$

Now substitute $x = 0, y' = 5, C_1 = 2$:

$$5 = 2(-2 - 0) + C_2(0 + 3) \Leftrightarrow 3C_2 = 9 \Leftrightarrow C_2 = 3.$$

Thus

$$y = 2e^{-2x} \cos(3x) + 3e^{-2x} \sin(3x).$$

5. [10 marks] Decide if each of the following infinite series converges or diverges. Mark your choice to the right, and justify your choice in the space provided.

(a) [3 marks] $\displaystyle\sum_{k=1}^{\infty} \frac{k}{\ln(k+5)}$ ○ Converges ⊗ Diverges

Justification:

$$\lim_{k\to\infty} \frac{k}{\ln(k+5)} = \lim_{k\to\infty} \frac{k+5}{1} = \infty \neq 0.$$

(b) [3 marks] $\displaystyle\sum_{m=1}^{\infty} \frac{13\cos^2 m}{m^3}$ ⊗ Converges ○ Diverges

Justification: by comparison with the p-series, with $p = 3$. Since

$$\frac{13\cos^2 m}{m^3} \leq \frac{13}{m^3}$$

and $\displaystyle\sum_{m=1}^{\infty} \frac{13}{m^3} = 13\sum_{m=1}^{\infty} \frac{1}{m^3}$ converges, so does $\displaystyle\sum_{m=1}^{\infty} \frac{13\cos^2 m}{m^3}$, by comparison.

(c) [4 marks] $\displaystyle\sum_{n=2}^{\infty} \frac{\ln n}{6\,e^{2n}}$ ⊗ Converges ○ Diverges

Justification: by the ratio test.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{e^{2n}\ln(n+1)}{e^{2n+2}\ln n} = \frac{1}{e^2}\lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{e^2} < 1.$$

Note: the root test works as well, but you have to establish

$$\lim_{n\to\infty} (\ln n)^{1/n} = 1,$$

which requires some work.

6. [10 marks] A tank with capacity 1500 L initially contains 400 L of water that is polluted with 30 kg of particulate matter. At $t = 0$, pure water is added at a rate of 20 L/min and the mixed solution is drained off at a rate of 10 L/min.

You may assume that if $x(t)$ is the mass of solute dissolved in a solution of volume $V(t)$, at time t, in a large mixing tank, then

$$\frac{d\,x(t)}{dt} + \frac{r_o\,x(t)}{V(t)} = r_i c_i,$$

where c_i is the concentration of solute in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

(a) [4 marks] Declare the values of r_o, r_i, c_i and $V(t)$ that you are using:

$$r_o = \underline{10}, \quad r_i = \underline{20}, \quad c_i = \underline{0}, \quad V(t) = \underline{400 + (r_i - r_o)\,t = 400 + 10t}$$

(b) [6 marks] How much particulate matter is in the tank when it reaches the point of overflowing?

Solution: since $c_i = 0$, the differential equation is separable.

$$\frac{dx}{dt} + \frac{10\,x}{400 + 10t} = 0 \Rightarrow \int \frac{dx}{x} = -\int \frac{dt}{40 + t} \Rightarrow \ln|x| = -\ln|40 + t| + C$$

We have $x_0 = 30 \Rightarrow C = \ln 30 + \ln 40 = \ln 1200$. The tank is full when

$$V(t) = 1500 \Leftrightarrow 400 + 10t = 1500 \Leftrightarrow t = 110.$$

When $t = 110$,

$$\ln x = -\ln(40 + 110) + \ln 1200 = \ln 8 \Rightarrow x = 8.$$

So there will be 8 kg of particulate matter in the tank when it reaches the point of overflowing.

Alternate Solution: The integrating factor of the differential equation

$$\frac{dx}{dt} + \frac{x}{40 + t} = 0$$

is $\mu = e^{\int \frac{dt}{40+t}} = e^{\ln(40+t)} = (40 + t)$, and so $x = C/\mu = C/(40 + t)$. We have $x_0 = 30 \Rightarrow C = 1200$. When $t = 110, x = 1200/150 = 8$, as before.

7. [8 marks; 4 for each part.] The parts of this question are unrelated.

(a) Find the exact sum of the infinite series $\displaystyle\sum_{k=0}^{\infty}\left(\frac{1}{3^k}-\frac{2}{5^{k+2}}\right)$

Solution: use the sum of an infinite geometric series, $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, if $|r|<1$.

$$\begin{aligned}
\sum_{k=0}^{\infty}\left(\frac{1}{3^k}-\frac{2}{5^{k+2}}\right) &= \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k - \frac{2}{5^2}\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k \\
&= \frac{1}{1-1/3} - \frac{2}{25}\cdot\frac{1}{1-1/5} \\
&= \frac{3}{2} - \frac{2}{25}\cdot\frac{5}{4} \\
&= \frac{3}{2} - \frac{1}{10} \\
&= \frac{7}{5}
\end{aligned}$$

(b) Does the infinite series $\displaystyle\sum_{n=0}^{\infty}\frac{4^{-n}+3^{-n}}{2^{-n}+3^{-n}}$ converge or diverge? Justify your answer.

Solution:

$$0 < a_n = \frac{4^{-n}+3^{-n}}{2^{-n}+3^{-n}} = \left(\frac{12^n}{12^n}\right)\left(\frac{4^{-n}+3^{-n}}{2^{-n}+3^{-n}}\right) = \frac{3^n+4^n}{6^n+4^n} < \frac{3^n+4^n}{6^n} = \frac{1}{2^n}+\frac{2^n}{3^n}$$

Since both

$$\sum_{n=0}^{\infty}\frac{1}{2^n} \text{ and } \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$$

are convergent infinite geometric series, the original series

$$\sum_{n=0}^{\infty}\frac{4^{-n}+3^{-n}}{2^{-n}+3^{-n}}$$

converges by the comparison test.

Note: the limit comparison test with $b_n = \left(\frac{2}{3}\right)^n$ works as well.