University of Toronto
Solutions to the **MAT187H1S TERM TEST**
of **Thursday, March 11, 2010**
Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Make sure this test contains 8 pages. Do not remove any pages from this test. Answer all questions. Present your solutions in the space provided. This means: show your work! The value for each question is indicated in parentheses beside the question number. **TOTAL MARKS: 60**

General Comments:

1. Questions 2, 3, 4, 5 and 7 on this test are very similar to homework problems. These questions should all have been aced!

2. Question 1 seems to have stumped a lot of people. But it was the simplest question on the test. No fancy techniques are needed: simply substitute y in the left side, and then solve for a, b and c.

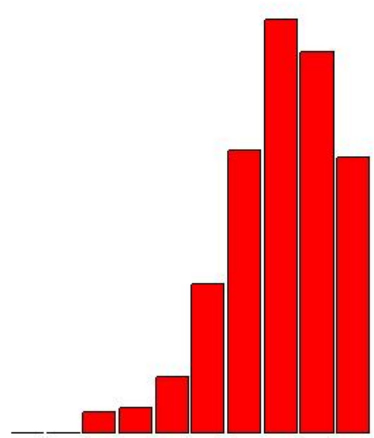
3. Question 6 is almost identical to Question 6 of last year's test.

4. All in all, this test was set with the hope of getting a high average; even with the poor showing on Question 1, it still seems to have worked out that way.

5. For Question 5(b) you could also have use the ratio test or the comparison test.

Breakdown of Results: 454 students wrote this test. The marks ranged from 20% to 100%, and the average was 74.4%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	17.2%
A	41.0%	80-89%	23.8%
B	25.8%	70-79%	25.8%
C	17.6%	60-69%	17.6%
D	9.3%	50-59%	9.3%
F	6.3%	40-49%	3.5%
		30-39%	1.5%
		20-29%	1.3%
		10-19%	0.0%
		0-9%	0.0%



1. [6 marks] Find values of a, b and c such that $y = ax^2 + bx + c$ is a solution to the differential equation

$$x^3 \frac{d^2y}{dx^2} + 4x^2 \frac{dy}{dx} + 7xy = 34x^3 - 22x^2 + 14x.$$

Solution:

$$\begin{aligned} y = ax^2 + bx + c \quad &\Rightarrow \quad \frac{dy}{dx} = 2ax + b \\ &\Rightarrow \quad \frac{d^2y}{dx^2} = 2a \end{aligned}$$

Substitute into the differential equation:

$$\begin{aligned} x^3 \frac{d^2y}{dx^2} + 4x^2 \frac{dy}{dx} + 7xy &= 34x^3 - 22x^2 + 14x \\ \Leftrightarrow x^3(2a) + 4x^2(2ax + b) + 7x(ax^2 + bx + c) &= 34x^3 - 22x^2 + 14x \\ \Leftrightarrow 17ax^3 + 11bx^2 + 7cx &= 34x^3 - 22x^2 + 14x \end{aligned}$$

So

$$17a = 34, \; 11b = -22 \;\text{ and } 7c = 14,$$

whence

$$a = 2, b = -2 \text{ and } c = 2.$$

2. [10 marks] Consider the initial value problem: $y' \cos^2 x = e^{-y} \sin x$ and $y(0) = 3$.

(a) [7 marks] Solve for y as an explicit function of x.

Solution: separate variables.

$$\begin{aligned} \frac{dy}{dx} \cos^2 x &= e^{-y} \sin x \\ \Leftrightarrow \int e^y \, dy &= \int \frac{\sin x}{\cos^2 x} \, dx = \int \tan x \sec x \, dx \\ \Leftrightarrow e^y &= \sec x + C \end{aligned}$$

To find C let $x = 0$ and $y = 3$:

$$e^3 = \sec 0 + C \Leftrightarrow C = e^3 - 1.$$

Thus

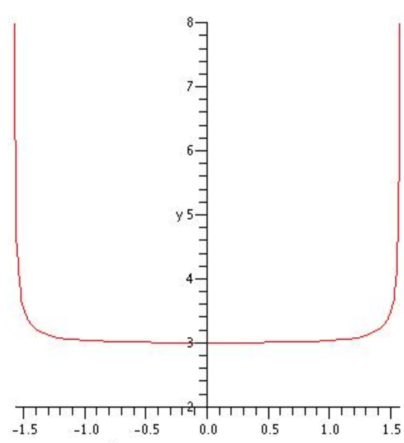
$$y = \ln(\sec x + e^3 - 1).$$

(b) [3 marks] Sketch the graph of y on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ by finding the intervals on which the graph of y is increasing or decreasing, and by finding the critical points of y.

Solution:

$$\frac{dy}{dx} = \frac{\sin x}{e^y \cos^2 x} \Rightarrow \begin{cases} \dfrac{dy}{dx} > 0 & \Leftrightarrow \quad \sin x > 0 \Rightarrow 0 < x < \dfrac{\pi}{2} \\ \dfrac{dy}{dx} = 0 & \Leftrightarrow \quad \sin x = 0 \Rightarrow x = 0 \\ \dfrac{dy}{dx} < 0 & \Leftrightarrow \quad \sin x < 0 \Rightarrow -\dfrac{\pi}{2} < x < 0 \end{cases}$$

So the graph looks like:



3. [8 marks] Find the general solution to the differential equation

$$\frac{dy}{dx} + y = \cos(e^x).$$

What is the particular solution that passes through the point $(\ln \pi, 2)$?

Solution: use the method of the integrating factor.

$$\mu = e^{\int dx} = e^x.$$

The general solution is

$$\begin{aligned} y &= \frac{\int \mu \cos(e^x)\, dx}{\mu} \\ &= \frac{\int e^x \cos(e^x)\, dx}{e^x} \\ &= e^{-x}\left(\sin(e^x) + C\right) \\ &= e^{-x}\sin(e^x) + Ce^{-x} \end{aligned}$$

Particular Solution: if $x = \ln \pi$ and $y = 2$, then

$$2 = e^{-\ln \pi}\sin(e^{\ln \pi}) + Ce^{-\ln \pi} \Rightarrow C = 2\pi.$$

So the particular solution passing through the point $(\ln \pi, 2)$ is

$$y = e^{-x}\sin(e^x) + 2\pi e^{-x} \text{ or } y = \frac{\sin(e^x) + 2\pi}{e^x}.$$

4. [9 marks] Solve for x as a function of t if $x'' + 2x' + 5x = 0$ and $x(0) = 2, x'(0) = -4$.

Solution: the auxiliary quadratic is $r^2 + 2r + 5$. Solve:

$$r^2 + 2r + 5 = 0 \Leftrightarrow r = \frac{-2 \pm \sqrt{4 - 20}}{2} = -1 \pm 2i.$$

Thus

$$x = C_1 e^{-t} \cos(2t) + C_2 e^{-t} \sin(2t).$$

To find C_1 use the initial condition $x = 2$ when $t = 0:$

$$2 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 2.$$

To find C_2 you need to find $x':$

$$x' = C_1(-e^{-t} \cos(2t) - 2e^{-t} \sin(2t)) + C_2(-e^{-t} \sin(2t) + 2e^{-t} \cos(2t)).$$

Now substitute $t = 0, x' = -4, C_1 = 2:$

$$-4 = 2(-1 - 0) + C_2(0 + 2) \Leftrightarrow 2C_2 = -2 \Leftrightarrow C_2 = -1.$$

Thus

$$x = 2e^{-t} \cos(2t) - e^{-t} \sin(2t).$$

5. [9 marks; 3 for each part.] Decide if the following infinite series converge or diverge. Summarize your work at the right by marking your choice, and by indicating which convergence/divergence test you are using.

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{4k^2 - 3k + 5}{8k^7 + 2k^3 - 8}$ $\otimes$ Converges ○ Diverges

by <u>the limit comparison test</u>

Calculation: use the fact that the p-series with $p = 5$ converges.

$$a_k = \frac{4k^2 - 3k + 5}{8k^7 + 2k^3 - 8}, \quad b_k = \frac{1}{k^5} \text{ and } \lim_{k \to \infty} \frac{a_k}{b_k} = \frac{1}{2}.$$

(b) $\displaystyle\sum_{m=1}^{\infty} \frac{m}{e^{m^2}}$ $\otimes$ Converges ○ Diverges

by <u>the integral test</u>

Calculation:

$$\int_1^{\infty} xe^{-x^2}\, dx = \frac{1}{2} \int_1^{\infty} e^{-u}\, du = \frac{1}{2} \left[-\frac{1}{e^u} \right]_1^{\infty} = \frac{1}{2e} < \infty.$$

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{4!\, n!\, 4^n}{(n+4)!}$ ○ Converges $\otimes$ Diverges

by <u>the ratio test</u>

Calculation:

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{4!(n+1)!4^{n+1}(n+4)!}{4!n!4^n(n+5)!} = \lim_{n \to \infty} \frac{4(n+1)}{n+5} = 4 > 1.$$

6. [10 marks] If x is the mass of salt dissolved in a saline solution of volume V, at time t, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_o}{V}x = r_i c_i,$$

where c_i is the concentration of salt in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A 150-liter tank initially contains 80 liters of brine (i.e. saline solution) containing 10 kg of salt. Brine containing 1 kg of salt per liter enters the tank at the rate of 5 liters per sec, and the well-mixed brine in the tank flows out at the rate of 3 liters per sec. How many kilograms of salt will the tank contain when it is full?

Solution: $V = 80 + (r_i - r_o)t = 80 + 2t$. The integrating factor of the differential equation

$$\frac{dx}{dt} + \frac{3}{80+2t}x = 5$$

is

$$\mu = e^{\int \frac{3}{80+2t}\,dt} = e^{3\ln(80+2t)/2} = (80+2t)^{3/2}$$

and so

$$x = \frac{\int 5\,\mu\,dt}{\mu} = \frac{(80+2t)^{5/2} + C}{(80+2t)^{3/2}} = (80+2t) + \frac{C}{(80+2t)^{3/2}}.$$

Use the initial condition $t = 0, x = 10$ to find C :

$$10 = 80 + \frac{C}{80^{3/2}} \Leftrightarrow C = -70 \cdot 80^{3/2} \simeq -50\,088.$$

Thus

$$x = (80+2t) - \frac{70 \cdot 80^{3/2}}{(80+2t)^{3/2}}.$$

The tank is full when $V = 150 \Leftrightarrow 80 + 2t = 150 \Leftrightarrow t = 35$. At $t = 35$,

$$x = (80+70) - \frac{70 \cdot 80^{3/2}}{(80+70)^{3/2}} = 150 - 70\left(\frac{8}{15}\right)^{3/2} \simeq 122.7.$$

So when the tank is full of brine it contains about 122.7 kilograms of salt.

7. [8 marks; 4 for each part.] Find the sum of the following infinite series:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{3}{9k^2+3k-2}$

Solution: use partial fractions.

$$\frac{3}{9k^2+3k-2} = \frac{1}{3k-1} - \frac{1}{3k+2}$$

So $\displaystyle\sum_{k=1}^{\infty} \frac{3}{9k^2+3k-2}$ is a telescoping sum:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{3}{9k^2+3k-2} &= \sum_{k=1}^{\infty} \left(\frac{1}{3k-1} - \frac{1}{3k+2} \right) \\
&= \frac{1}{2} - \frac{1}{5} + \frac{1}{5} - \frac{1}{8} + \frac{1}{8} - \frac{1}{11} + \frac{1}{11} - \frac{1}{14} + \cdots \\
&= \frac{1}{2}
\end{aligned}$$

More formally:

$$S_N = \frac{1}{2} - \frac{1}{3N+2} \Rightarrow S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{3N+2} \right) = \frac{1}{2}.$$

(b) $\displaystyle\sum_{k=0}^{\infty} \left(\frac{1}{2^k} - \frac{3}{5^{k+1}} \right)$

Solution: use the sum of an infinite geometric series, $\displaystyle\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$, if $|r| < 1$.

$$\begin{aligned}
\sum_{k=0}^{\infty} \left(\frac{1}{2^k} - \frac{3}{5^{k+1}} \right) &= \sum_{k=0}^{\infty} \left(\frac{1}{2} \right)^k - \frac{3}{5} \sum_{k=0}^{\infty} \left(\frac{1}{5} \right)^k \\
&= \frac{1}{1-1/2} - \frac{3}{5} \frac{1}{1-1/5} \\
&= 2 - \frac{3}{5} \cdot \frac{5}{4} \\
&= 2 - \frac{3}{4} \\
&= \frac{5}{4}
\end{aligned}$$