University of Toronto
Solutions to the **MAT187H1S TERM TEST**
of **Thursday, March 12, 2009**
Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

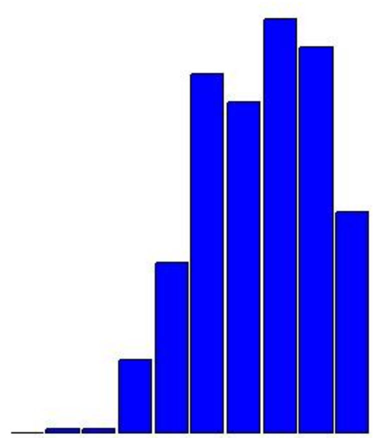
Instructions: Make sure this test contains 8 pages. Do not remove any pages from this test. Answer all questions. Present your solutions in the space provided. This means: show your work! The value for each question is indicated in parentheses beside the question number. **TOTAL MARKS: 60**

Comments and Results:

1. Questions 1, 2 and 4 are completely routine; everyone should have aced them.
2. Question 3 can be simplified in many ways; the given parametric equations describe a circle.
3. To set things up in Question 5 you need to use similar triangles.
4. Question 6 was almost an exact copy of last year's question 6; sorry about that!
5. Question 7 was right out of the homework from Section 8.5; see #3 on page 628. Two marks were deducted if you didn't handle the absolute value signs correctly, and three marks were deducted if you didn't solve for y. Finally, you can't let $y = \sin\theta$ in this question since then $|y| \leq 1$, contradicting the initial condition $y = 3$ if $x = 0$. You could let $y = \sec\theta$, instead of using partial fractions.

Breakdown of Results: 427 students wrote this test. The marks ranged from 11.7% to 100%, and the average was 68.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	11.2%
A	30.9%	80-89%	19.7%
B	21.1 %	70-79%	21.1%
C	16.9%	60-69%	16.9%
D	18.3%	50-59%	18.3%
F	12.9%	40-49%	8.7%
		30-39%	3.7%
		20-29%	0.2%
		10-19%	0.2%
		0-9%	0.0 %



1. [8 marks] Consider the curve with parametric equations

$$x = t^3; y = t^2 + \ln t.$$

Find both

$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}$$

at the point on the curve at which $t = 1$.

Solution: for $t \neq 0$:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t + 1/t}{3t^2} = \frac{2}{3t} + \frac{1}{3t^3}$$

and

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{2}{3t} + \frac{1}{3t^3}\right)}{3t^2} = -\frac{\left(\frac{2}{3t^2} + \frac{1}{t^4}\right)}{3t^2} = -\frac{2}{9t^4} - \frac{1}{3t^6}$$

So at $t = 1$,

$$\frac{dy}{dx} = \frac{2}{3} + \frac{1}{3} = 1$$

and

$$\frac{d^2y}{dx^2} = -\frac{2}{9} - \frac{1}{3} = -\frac{5}{9}.$$

2. [10 marks] Solve for $x(t)$ as a function of t if

$$x''(t) + 12x'(t) + 100x(t) = 0; \quad x(0) = 12 \text{ and } x'(0) = -32.$$

Sketch the graph of $x(t)$ for $0 \leq t \leq \pi/2$, and find the pseudo period and the time-varying amplitude of $x(t)$.

Solution: the auxiliary quadratic is $r^2 + 12r + 100$. Solve:

$$r^2 + 12r + 100 = 0 \Leftrightarrow r = \frac{-12 \pm \sqrt{144 - 400}}{2} = -6 \pm 8i.$$

Thus

$$x = C_1 e^{-6t} \cos(8t) + C_2 e^{-6t} \sin(8t).$$

To find C_1 use the initial condition $x = 12$ when $t = 0$:

$$12 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 12.$$

To find C_2 you need to find x' :

$$x' = C_1(-6e^{-6t}\cos(8t) - 8e^{-6t}\sin(8t)) + C_2(-6e^{-6t}\sin(8t) + 8e^{-6t}\cos(8t)).$$

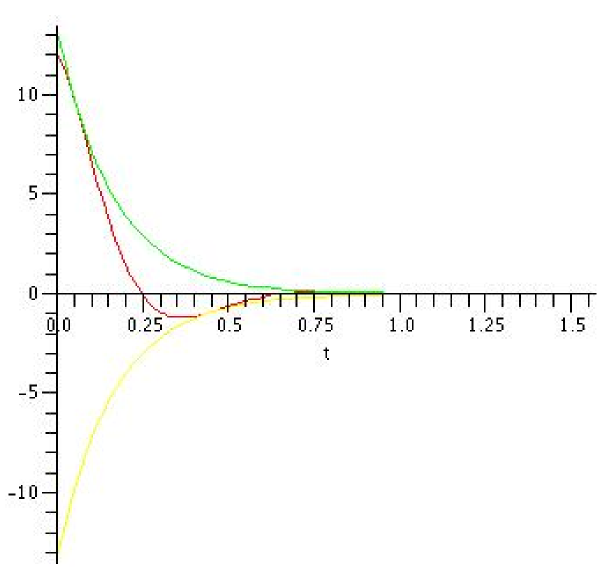
Now substitute $t = 0, x' = -32, C_1 = 12$:

$$-32 = 12(-6 - 0) + C_2(0 + 8) \Leftrightarrow 8C_2 = 42 \Leftrightarrow C_2 = 5.$$

Thus

$$x = 12e^{-6t}\cos(8t) + 5e^{-6t}\sin(8t) = 13e^{-6t}\cos\left(8t - \tan^{-1}\left(\frac{5}{12}\right)\right).$$

Graph:



The **pseudo period** is

$$\frac{2\pi}{8} = \frac{\pi}{4};$$

time-varying amplitude is

$$\sqrt{12^2 + 5^2}e^{-6t} = 13e^{-6t}.$$

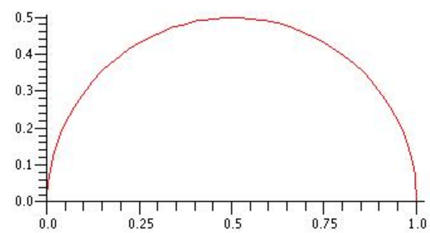
3. [8 marks] Consider the curve with parametric equations

$$x = \sin^2 t, y = \sin t \cos t, \text{ for } 0 \leq t \leq \frac{\pi}{2}.$$

Sketch this curve and compute its arc length.

Solution: $t = 0 \Rightarrow (x, y) = (0, 0); t = \pi/2 \Rightarrow (x, y) = (1, 0)$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{y'(t)}{x'(t)} \\ &= \frac{\cos^2 t - \sin^2 t}{2 \sin t \cos t} \\ \text{(optional)} \quad &= \frac{\cos(2t)}{\sin(2t)} \end{aligned}$$



$\frac{dy}{dx} = 0 \Rightarrow t = \frac{\pi}{4} \Rightarrow (x, y) = \left(\frac{1}{2}, \frac{1}{2}\right)$, which is the maximum point on the graph.

Arc Length:

$$\begin{aligned} L &= \int_0^{\pi/2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_0^{\pi/2} \sqrt{(2 \sin t \cos t)^2 + (\cos^2 t - \sin^2 t)^2}\, dt \\ &= \int_0^{\pi/2} \sqrt{4 \sin^2 t \cos^2 t + \cos^4 t - 2 \sin^2 t \cos^2 t + \sin^4 t}\, dt \\ &= \int_0^{\pi/2} \sqrt{\cos^4 t + 2 \sin^2 t \cos^2 t + \sin^4 t}\, dt \\ &= \int_0^{\pi/2} \sqrt{(\cos^2 t + \sin^2 t)^2}\, dt \\ &= \int_0^{\pi/2} dt = \frac{\pi}{2} \end{aligned}$$

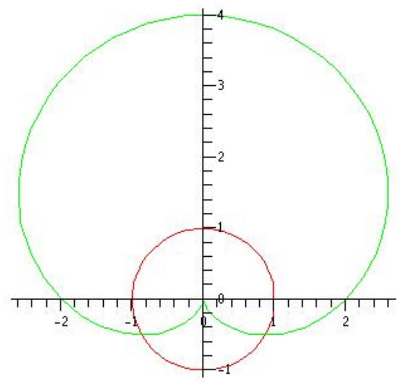
Easier Calculation: $L = \int_0^{\pi/2} \sqrt{\sin^2(2t) + \cos^2(2t)}\, dt = \int_0^{\pi/2} dt = \frac{\pi}{2}$.

Alternate Solution: $r^2 = x^2 + y^2 = \sin^4 t + \sin^2 t \cos^2 t = \sin^2 t \Rightarrow r = \sin t$, which is the polar equation of a circle with centre $(1/2, 0)$ and radius $1/2$; half of its circumference is $\pi/2$.

4. [8 marks] Find the area of the region within the cardioid with polar equation $r = 2 + 2\sin\theta$ but outside the circle with polar equation $r = 1$.

Solution: find the intersection points.

$$\begin{aligned} & 2 + 2\sin\theta = 1 \\ \Rightarrow\ & \sin\theta = -\frac{1}{2} \\ \Rightarrow\ & \theta = -\frac{\pi}{6} \text{ or } \frac{7\pi}{6} \end{aligned}$$



Set Up:

$$A = \frac{1}{2}\int_{-\pi/6}^{7\pi/6} \left((2 + 2\sin\theta)^2 - 1^2\right)\, d\theta \text{ or } A = \int_{-\pi/6}^{\pi/2} \left((2 + 2\sin\theta)^2 - 1^2\right)\, d\theta.$$

Calculation:

$$\begin{aligned} A &= \int_{-\pi/6}^{\pi/2} \left((2 + 2\sin\theta)^2 - 1^2\right)\, d\theta. \\ &= \int_{-\pi/6}^{\pi/2} \left(3 + 8\sin\theta + 4\sin^2\theta\right)\, d\theta \\ &= \int_{-\pi/6}^{\pi/2} \left(3 + 8\sin\theta + 2(1 - \cos(2\theta))\right)\, d\theta \\ &= \int_{-\pi/6}^{\pi/2} \left(5 + 8\sin\theta - 2\cos(2\theta)\right)\, d\theta \\ &= \left[5\theta - 8\cos\theta - \sin(2\theta)\right]_{-\pi/6}^{\pi/2} \\ &= \frac{5\pi}{2} - 0 - 0 + \frac{5\pi}{6} + 4\sqrt{3} - \frac{\sqrt{3}}{2} \\ &= \frac{10\pi}{3} + \frac{7\sqrt{3}}{2} \end{aligned}$$

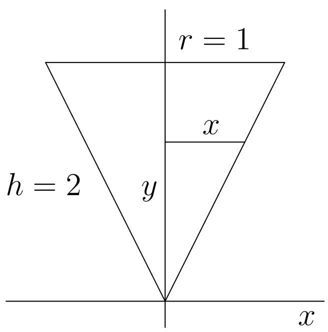
5. [8 marks] Torricelli's Law states that

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy},$$

where y is the depth of a fluid in a tank at time t, $A(y)$ is the cross-sectional area of the tank at height y above the exit hole, a is the cross-sectional area of the exit hole, and g is the acceleration due to gravity.

A water tank is in the shape of an inverted cone. Its height is 2 m and the radius of its circular top is 1 m. The tank is initially full of water. At 12 noon a plug at the bottom of the tank is removed, and 20 min later the depth of water in the tank is 1 m. At what time will the tank be completely empty?

Solution:



By similar triangles

$$\begin{aligned} \frac{x}{y} &= \frac{1}{2} \\ \Rightarrow x &= \frac{y}{2} \end{aligned}$$

So

$$A(y) = \pi x^2 = \frac{\pi y^2}{4}.$$

Solve the DE by separating variables:

$$\begin{aligned} A(y)\frac{dy}{dt} = -a\sqrt{2gy} \quad &\Leftrightarrow \quad \pi \int \frac{y^2}{4\sqrt{y}}\,dy = -\pi \int K\,dt, \text{ for } K = \frac{a\sqrt{2g}}{\pi} \\ &\Leftrightarrow \quad \frac{1}{4}\int \left(y^{3/2}\right)\,dy = -\int K\,dt \\ &\Leftrightarrow \quad \frac{y^{5/2}}{10} = -Kt + C, \text{ for some } C \end{aligned}$$

Let t be measured in minutes; let $t = 0$ be noon. When $t = 0, y = 2$, so

$$\frac{2^{5/2}}{10} = C \Leftrightarrow C = \frac{2\sqrt{2}}{5}.$$

When $t = 20, y = 1$, so

$$\frac{1}{10} = -20K + \frac{2\sqrt{2}}{5} \Leftrightarrow 20K = \frac{2\sqrt{2}}{5} - \frac{1}{10} \Leftrightarrow K = \frac{4\sqrt{2}-1}{200}$$

The tank is empty when $y = 0$:

$$0 = -Kt + C \Leftrightarrow t = \frac{C}{K} = \frac{80\sqrt{2}}{4\sqrt{2}-1} \simeq 24.3.$$

So the tank will be empty at about 12:24 PM.

6. [10 marks] If x is the mass of salt dissolved in a saline solution of volume V, at time t, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_o}{V}x = r_i c_i,$$

where c_i is the concentration of salt in a solution entering the mixing tank at rate r_i, and r_o is the rate at which the well-mixed solution is leaving the tank.

A 150-liter tank initially contains 100 liters of brine (i.e. saline solution) containing 20 kg of salt. Brine containing 1 kg of salt per liter enters the tank at the rate of 5 liters per sec, and the well-mixed brine in the tank flows out at the rate of 3 liters per sec. How many kilograms of salt will the tank contain when it is full?

Solution: $V = 100 + (r_i - r_o)t = 100 + 2t$. The integrating factor of the differential equation

$$\frac{dx}{dt} + \frac{3}{100+2t}x = 5$$

is

$$\rho = e^{\int \frac{3}{100+2t}\,dt} = e^{3\ln(100+2t)/2} = (100+2t)^{3/2}$$

and so

$$x = \frac{\int 5\,\rho\,dt}{\rho} = \frac{(100+2t)^{5/2} + C}{(100+2t)^{3/2}} = (100+2t) + \frac{C}{(100+2t)^{3/2}}.$$

Use the initial condition $t = 0, x = 20$ to find C :

$$20 = 100 + \frac{C}{100^{3/2}} \Leftrightarrow C = -80\,000.$$

Thus

$$x = (100+2t) - \frac{80\,000}{(100+2t)^{3/2}}.$$

The tank is full when $V = 150 \Leftrightarrow 100 + 2t = 150 \Leftrightarrow t = 25$. At $t = 25$,

$$x = (100+50) - \frac{80\,000}{(100+50)^{3/2}} = 150 - \frac{32}{9}\sqrt{150} \simeq 106.45.$$

So when the tank is full of brine it contains about 106.45 kilograms of salt.

7. [8 marks] Solve for y as a function of x if

$$\frac{dy}{dx} = 1 - y^2 \text{ and } y = 3 \text{ if } x = 0.$$

Soluiton: separate variables.

$$\int \frac{1}{1-y^2}\,dy = \int dx \quad \Rightarrow \quad \int \frac{1}{(1+y)(1-y)}\,dy = x + c$$

$$\text{(partial fractions)} \quad \Rightarrow \quad \frac{1}{2}\int \left(\frac{1}{1+y} + \frac{1}{1-y}\right) dy = x + c$$

$$\Rightarrow \quad \frac{1}{2}\left(\ln|1+y| - \ln|1-y|\right) = x + c$$

$$\Rightarrow \quad \ln\left|\frac{1+y}{1-y}\right| = 2x + c'$$

To find c', substitute $x = 0$ and $y = 3$:

$$c' = \ln\left|\frac{1+3}{1-3}\right| = \ln 2.$$

Now solve for y :

$$\ln\left|\frac{1+y}{1-y}\right| = 2x + \ln 2 \quad \Rightarrow \quad \left|\frac{1+y}{1-y}\right| = e^{2x+\ln 2} = 2e^{2x}$$

$$\Rightarrow \quad \frac{1+y}{1-y} = -2e^{2x}, \text{ since } y = 3, \text{ if } x = 0$$

$$\Rightarrow \quad 1 + y = -2e^{2x} + 2ye^{2x}$$

$$\Rightarrow \quad y(1 - 2e^{2x}) = -1 - 2e^{2x}$$

$$\Rightarrow \quad y = \frac{1 + 2e^{2x}}{2e^{2x} - 1}$$