University of Toronto
Solutions to **MAT187H1S TERM TEST**
of **Thursday, March 20, 2008**
Duration: 90 minutes

Only aids permitted: Casio 260, Sharp 520, or Texas Instrument 30 calculator.

Instructions: Make sure this test has 8 pages. Answer all questions. Present your solutions in the space provided. The value for each question is indicated in parantheses beside the question number.

TOTAL MARKS: 60

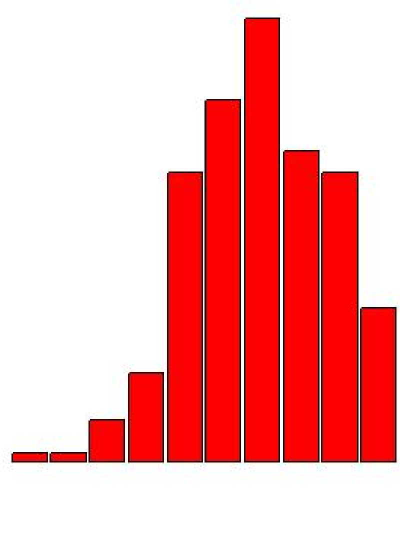
Comments and Results: The marks on this test were on average much lower than on the first test. Nevertheless, the questions on this test are all very routine; four of them were right out of the homework. Nobody – nobody who did their homework! – should have failed this test.
Concerning Question 5, part (b): you can't use the second derivative test to show that $x = 200$ is a maximum since

$$\begin{aligned}\frac{dx}{dt} = (0.8)x - (0.004)x^2 \quad &\Rightarrow \quad \frac{d^2x}{dt^2} = (0.8)\frac{dx}{dt} - (0.008)x\frac{dx}{dt} \\ &\Rightarrow \quad \frac{d^2x}{dt^2} = 0 \text{ at the critical value } x = 200,\end{aligned}$$

which means the second derivative test doesn't apply.

Breakdown of Results: 469 students wrote this test. The marks ranged from 7% to 100%, and the average was 63.5%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	7.7%
A	22.2%	80-89%	14.5%
B	15.6%	70-79%	15.6%
C	22.2%	60-69%	22.2%
D	18.1%	50-59%	18.1%
F	21.9%	40-49%	13.0%
		30-39%	6.0%
		20-29%	2.1%
		10-19%	0.4%
		0-9%	0.4 %



1. [8 marks] Consider the curve with parametric equations

$$x = t^2 + 4t; y = t^3 - 3t.$$

Find both

$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}$$

in terms of the parameter t.

Solution:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3t^2 - 3}{2t + 4}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{dy'}{dt}}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{3t^2 - 3}{2t + 4}\right)}{2t + 4} = \frac{\frac{6t(2t + 4) - 2(3t^2 - 3)}{(2t + 4)^2}}{2t + 4} = \frac{6t^2 + 24t + 6}{(2t + 4)^3}$$

2. [8 marks] Solve for y as a function of x if

$$y'' + 4y' + 20\,y = 0;\; y(0) = 9,\; y'(0) = 10.$$

Solution: the auxiliary quadratic is $r^2 + 4r + 20$. Solve:

$$r^2 + 4r + 20 = 0 \Leftrightarrow r = \frac{-4 \pm \sqrt{16 - 80}}{2} = -2 \pm 4i.$$

Thus

$$y = C_1 e^{-2x} \cos(4x) + C_2 e^{-2x} \sin(4x).$$

To find C_1 use the initial condition $y = 9$ when $x = 0$:

$$9 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 \Leftrightarrow C_1 = 9.$$

To find C_2 you need to find y' :

$$y' = C_1(-2e^{-2x} \cos(4x) - 4e^{-2x} \sin(4x)) + C_2(-2e^{-2x} \sin(4x) + 4e^{-2x} \cos(4x)).$$

Now substitute $x = 0, y' = 10, C_1 = 9$:

$$10 = 9(-2 - 0) + C_2(0 + 4) \Leftrightarrow 4C_2 = 28 \Leftrightarrow C_2 = 7.$$

Thus

$$y = 9e^{-2x} \cos(4x) + 7e^{-2x} \sin(4x).$$

3. [8 marks] Consider the curve with parametric equations

$$x = \sin t, y = \sin t \cos t, \text{ for } 0 \leq t \leq \frac{\pi}{2}.$$

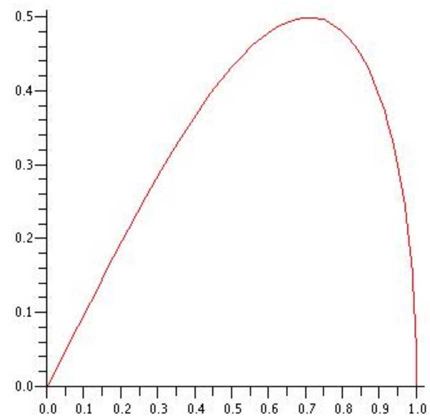
Plot this curve and compute the volume of the solid of revolution produced by revolving the curve around the x-axis.

Solution:

$t = 0 \Rightarrow (x, y) = (0, 0)$.
$t = \pi/2 \Rightarrow (x, y) = (1, 0)$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos^2 t - \sin^2 t}{\cos t} = 0 \\ \Rightarrow t &= \frac{\pi}{4} \end{aligned}$$

$t = \pi/4 \Rightarrow (x, y) = (1/\sqrt{2}, 1/2)$; which is the maximum point.



$$\begin{aligned} V &= \int_0^1 \pi y^2 \, dx \\ &= \pi \int_0^{\pi/2} \sin^2 t \cos^2 t \, \cos t \, dt \\ &= \pi \int_0^{\pi/2} \sin^2 t \, (1 - \sin^2 t) \, \cos t \, dt \\ (\text{Let } u = \sin t) &= \pi \int_0^1 u^2 (1 - u^2) \, du \\ &= \pi \int_0^1 \left(u^2 - u^4\right) \, du \\ &= \pi \left[\frac{1}{3} u^3 - \frac{1}{5} u^5\right]_0^1 \\ &= \pi \left(\frac{1}{3} - \frac{1}{5}\right) \\ &= \frac{2\pi}{15} \end{aligned}$$

Alternate solution: eliminate the parameter: $y = x\sqrt{1 - x^2}, 0 \leq x \leq 1$. So

$$V = \int_0^1 \pi y^2 \, dx = \pi \int_0^1 x^2 (1 - x^2) \, dx = \frac{2\pi}{15}.$$

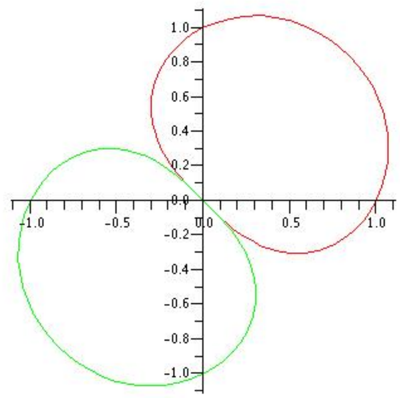
4. [8 marks] Plot the polar curve with polar equation $r^2 = \cos\theta + \sin\theta$ for

$$-\frac{\pi}{4} \leq \theta \leq \frac{3\pi}{4}$$

and find the area within the curve.

Solution:

$\theta = -\pi/4 \Rightarrow r^2 = 0$
$\theta = 0 \Rightarrow r^2 = 1$
$\theta = \pi/4 \Rightarrow r^2 = \sqrt{2}$
$\theta = \pi/2 \Rightarrow r^2 = 1$
$\theta = 3\pi/4 \Rightarrow r^2 = 0$



$$\begin{aligned} r^2 &= \cos\theta + \sin\theta \\ \Rightarrow r &= \pm\sqrt{\cos\theta + \sin\theta} \end{aligned}$$

$r > 0$ is the red curve; $r < 0$ is the green curve.

$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_{-\pi/4}^{3\pi/4} r^2\,d\theta\right) \\ &= \int_{-\pi/4}^{3\pi/4} (\cos\theta + \sin\theta)\,d\theta \\ &= [\sin\theta - \cos\theta]_{-\pi/4}^{3\pi/4} \\ &= \sin\left(\frac{3\pi}{4}\right) - \cos\left(\frac{3\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right) + \cos\left(-\frac{\pi}{4}\right) \\ &= \frac{4}{\sqrt{2}} \\ &= 2\sqrt{2} \end{aligned}$$

5. [10 marks] As the salt KNO_3 dissolves in methanol, the number $x(t)$ of grams of the salt in solution after t seconds satisfies the differential equation

$$\frac{dx}{dt} = (0.8)x - (0.004)x^2.$$

(a) [6 marks] If $x = 50$ when $t = 0$, how long will it take an additional 50 g of salt to dissolve?

Solution: this is logistic growth.

$$\begin{aligned} \frac{dx}{dt} &= (0.8)x - (0.004)x^2 = 0.8\,x\left(1 - \frac{x}{200}\right) \\ \Rightarrow x &= \frac{200}{1 + Ae^{-0.8t}}, \text{ for some constant } A \end{aligned}$$

Use the initial condition $t = 0, x = 50$ to find A :

$$50 = \frac{200}{1 + A} \Leftrightarrow 1 + A = 4 \Leftrightarrow A = 3.$$

Now let $x = 100$ and solve for t :

$$100 = \frac{200}{1 + 3e^{-0.8t}} \Leftrightarrow 3e^{-0.8t} = 1 \Leftrightarrow e^{0.8t} = 3 \Leftrightarrow t = \frac{\ln 3}{0.8} \simeq 1.37$$

So it will take about 1.37 seconds for another 50 grams of KNO_3 to dissolve.

(b) [4 marks] What is the maximum amount of salt that will ever dissolve in the methanol?

Solution: find limit of x as $t \to \infty$:

$$\lim_{t\to\infty} \frac{200}{1 + 3e^{-0.8t}} = \frac{200}{1 + 0} = 200.$$

Alternate Calculations: if you didn't remember the logistic equation.

$$\begin{aligned} &\frac{dx}{dt} = 0.8\,x\left(1 - \frac{x}{200}\right) \Rightarrow \int \left(\frac{200}{x(200 - x)}\right) dx = \int 0.8\,dt \\ \Rightarrow\ & \int \left(\frac{1}{x} + \frac{1}{200 - x}\right) dx = 0.8t + C \Rightarrow \ln|x| - \ln|200 - x| = 0.8t + C \\ \Rightarrow\ & \ln\left|\frac{x}{200 - x}\right| = 0.8t + C \Rightarrow \frac{x}{200 - x} = \pm Be^{0.8t} \Rightarrow \frac{200 - x}{x} = Ae^{-0.8t} \\ \Rightarrow\ & \frac{200}{x} = 1 + Ae^{-0.8t} \Rightarrow x = \frac{200}{1 + Ae^{-0.8t}}; \text{ with } A = \pm B^{-1} = \pm e^{-C} \end{aligned}$$

Now proceed as above.

6. [10 marks] If x is the amount of salt disolved in a saline solution of volume V, at time t, in a large mixing tank, then

$$\frac{dx}{dt} + \frac{r_0}{V}x = r_i c_i,$$

where c_i is the concentration of salt in a solution entering the mixing tank at rate r_i, and r_0 is the rate at which the well-mixed solution is leaving the tank.

A 200-liter tank initially contains 100 liters of brine (i.e. saline solution) containing 25 kg of salt. Brine containing 1 kg of salt per liter enters the tank at the rate of 5 liters per sec, and the well-mixed brine in the tank flows out at the rate of 3 liters per sec. How much salt will the tank contain when it is full of brine?

Solution: $V = 100 + (r_i - r_o)t = 100 + 2t$. The integrating factor of the differential equation

$$\frac{dx}{dt} + \frac{3}{100+2t}x = 5$$

is

$$\rho = e^{\int \frac{3}{100+2t}\,dt} = e^{3\ln(100+2t)/2} = (100+2t)^{3/2}$$

and so

$$x = \frac{\int 5\,\rho\,dt}{\rho} = \frac{(100+2t)^{5/2} + C}{(100+2t)^{3/2}} = (100+2t) + \frac{C}{(100+2t)^{3/2}}.$$

Use the initial condition $t = 0, x = 25$ to find C :

$$25 = 100 + \frac{C}{100^{3/2}} \Leftrightarrow C = -75\,000.$$

Thus

$$x = (100+2t) - \frac{75\,000}{(100+2t)^{3/2}}.$$

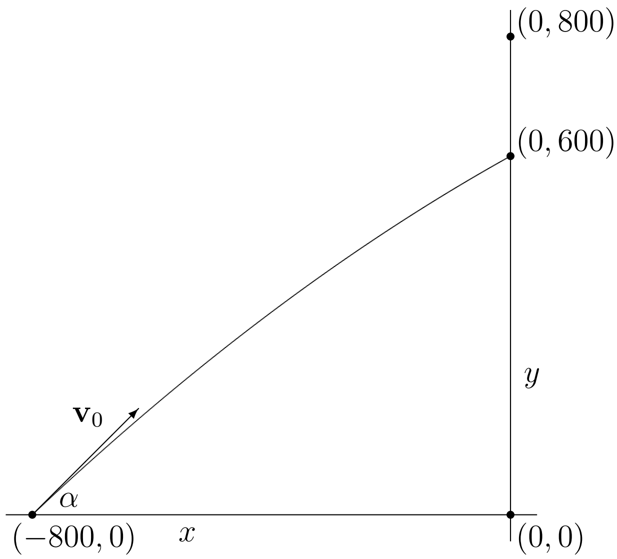
The tank is full when $V = 200 \Leftrightarrow 100 + 2t = 200 \Leftrightarrow t = 50$. At $t = 50$,

$$x = (100+100) - \frac{75\,000}{(100+100)^{3/2}} = 200 - \frac{75}{2\sqrt{2}} \simeq 173.5.$$

So when the tank is full of brine it contains about 173.5 kilograms of salt.

7. [8 marks] A bomb is dropped (initial speed zero) from a helicopter hovering at a height 800 m. A projectile is fired from a gun located on the ground 800 m from the point directly beneath the helicopter. The projectile is supposed to intercept the bomb at a height of exactly 600 m. If the projectile is fired at the same instant that the bomb is dropped, what should be its initial speed and angle of inclination? (Use $g = 9.8$ m/sec^2.)

Solution:



Trajectory of the bomb is

$$y = 800 - \frac{1}{2}gt^2.$$

So

$$\begin{aligned} y = 600 \quad &\Rightarrow \quad -200 = -\frac{1}{2}gt^2 \\ &\Rightarrow \quad t^2 = \frac{400}{g} \\ &\Rightarrow \quad t = \frac{20}{\sqrt{g}} \end{aligned}$$

The parametric equations of the projectile's trajectory are

$$x = -800 + (v_0 \cos\alpha)t; y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2.$$

At $t = \dfrac{20}{\sqrt{g}}$,

$$\begin{aligned} (x, y) = (0, 600) \quad &\Rightarrow \quad \begin{cases} -800 + \dfrac{20v_0 \cos\alpha}{\sqrt{g}} &= 0 \\ -200 + \dfrac{20v_0 \sin\alpha}{\sqrt{g}} &= 600 \end{cases} \\ &\Rightarrow \quad \begin{cases} v_0 \cos\alpha &= 40\sqrt{g} \\ v_0 \sin\alpha &= 40\sqrt{g} \end{cases} \\ &\Rightarrow \quad \tan\alpha = 1 \Rightarrow \alpha = 45^\circ \\ &\Rightarrow \quad v_0 = \frac{40\sqrt{g}}{\cos 45^\circ} = 40\sqrt{2g} \simeq 177.1 \end{aligned}$$

So the projectile should be aimed at an angle of 45° from the horizontal with an initial speed of 177.1 m/sec.
NB: if the projectile is fired from the right side of the helicopter, then $\alpha = 135^\circ$ counter clockwise from the positive x-axis.