University of Toronto
Faculty of Applied Science and Engineering
Solutions to Final Examination, June 2017
Duration: 2 and 1/2 hrs
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT187H1F - Calculus II

Examiner: D. Burbulla

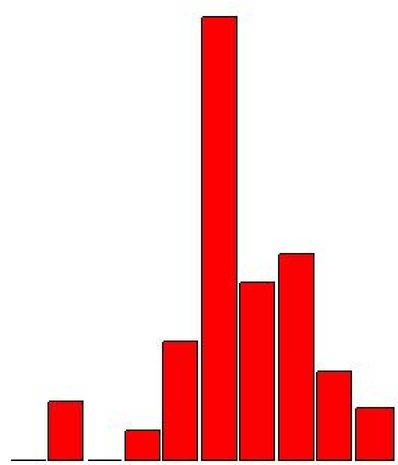
Exam Type: A. Aids permitted: Formula Sheet, and Casio FX-991 or Sharp EL-520 calculator.

General Comments:

1. Only Questions 3 and 5 had a failing average; although Question 3 was basically a repeat of a similar question on Test 2, and Question 5—which is not hard—was done in class.
2. Question 4 was a very easy polar area question, but many students made it unnecessarily complicated.
3. Question 6 was almost identical to a question on the June 2016 exam; it should have been aced.

Breakdown of Results: 40 students wrote this exam. The marks ranged from 12% to 91%, and the average was 59.75%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	5.0%
A	12.5%	80-89%	7.5%
B	17.5%	70-79%	17.5%
C	15.0%	60-69%	15.0%
D	37.5%	50-59%	37.5%
F	17.5%	40-49%	10.0 %
		30-39%	2.5%
		20-29%	0.0%
		10-19%	5.0%
		0-9%	0.0%



1. [30 marks; avg: 22.8] Consider the curve in space with vector equation

$$\mathbf{r}(t) = \left\langle \cos t + t\sin t, \sin t - t\cos t, \frac{\sqrt{3}\,t^2}{2} \right\rangle,$$

for $t \geq 0$.

Average this page: 11.6/13.

(a) [9 marks] Calculate and simplify the following:

(*i*) [3 marks] $\mathbf{r}'(t)$

Solution:

$$\mathbf{r}'(t) = \left\langle -\sin t + \sin t + t\cos t, \cos t - \cos t + t\sin t, \sqrt{3}\,t \right\rangle = \left\langle t\cos t, t\sin t, \sqrt{3}\,t \right\rangle$$

(*ii*) [3 marks] $\mathbf{r}''(t)$

Solution: differentiate your answer from part (a).

$$\mathbf{r}''(t) = \left\langle \cos t - t\sin t, \sin t + t\cos t, \sqrt{3} \right\rangle$$

(*iii*) [3 marks] $\mathbf{r}'''(t)$

Solution: differentiate your answer from part (b).

$$\mathbf{r}'''(t) = \langle -\sin t - \sin t - t\cos t, \cos t + \cos t - t\sin t, 0\rangle = \langle -2\sin t - t\cos t, 2\cos t - t\sin t, 0\rangle$$

(b) [4 marks] What is the length of the curve for $0 \leq t \leq 3$?

Solution: use the length formula.

$$L = \int_0^3 |\mathbf{r}'(t)|\,dt = \int_0^3 \sqrt{t^2 + 3t^2}\,dt = \int_0^3 2t\,dt = \left[t^2\right]_0^3 = 9.$$

(c) [8 marks; avg: 6.2] Find $\mathbf{T}(\mathbf{t}), \mathbf{N}(t)$ and $\mathbf{B}(t)$, the **unit** tangent, normal and binormal vectors, respectively, of the curve.

Solution: for $t > 0$.

$$\begin{aligned}
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} &= \frac{\left\langle t\cos t, t\sin t, \sqrt{3}\,t\right\rangle}{\left|\left\langle t\cos t, t\sin t, \sqrt{3}\,t\right\rangle\right|} \\
&= \frac{\left\langle t\cos t, t\sin t, \sqrt{3}\,t\right\rangle}{2t} \\
&= \frac{1}{2}\left\langle \cos t, \sin t, \sqrt{3}\right\rangle
\end{aligned}$$

$$\begin{aligned}
\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} &= \frac{\frac{1}{2}\langle -\sin t, \cos t, 0\rangle}{\frac{1}{2}|\langle -\sin t, \cos t, 0\rangle|} \\
&= \langle -\sin t, \cos t, 0\rangle
\end{aligned}$$

$$\begin{aligned}
\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) &= \frac{1}{2}\left\langle \cos t, \sin t, \sqrt{3}\right\rangle \times \langle -\sin t, \cos t, 0\rangle \\
&= \frac{1}{2}\left\langle -\sqrt{3}\,\cos t, -\sqrt{3}\,\sin t, 1\right\rangle
\end{aligned}$$

Continued...

(d) [9 marks; avg: 5.0] Show that for this curve $\tau = \sqrt{3}\,\kappa$ for all $t > 0$, where κ is the curvature of the curve and τ is its torsion.

Solution: the shortest way, making use of your previous calculations, would be

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1/2}{2t} = \frac{1}{4t}$$

and

$$\tau = -\frac{d\mathbf{B}}{ds} \cdot \mathbf{N} = -\frac{\mathbf{B}'(t) \cdot \mathbf{N}(\mathbf{t})}{|\mathbf{r}'(t)|} = -\frac{1}{2t}\left\langle \frac{\sqrt{3}}{2}\sin t, -\frac{\sqrt{3}}{2}\cos t, 0 \right\rangle \cdot \langle -\sin t, \cos t, 0\rangle = \frac{\sqrt{3}}{4t};$$

so, YES, $\tau = \sqrt{3}\,\kappa$.

OR: looking at the formulas for κ and τ supplied on the Formula Sheet, you can see that both can be calculated in terms of $\mathbf{r}'(t) \times \mathbf{r}''(t)$. Using the calculations from part (a) we have

$$\begin{aligned} \mathbf{r}'(t) \times \mathbf{r}''(t) &= \left\langle t\cos t, t\sin t, \sqrt{3}\,t \right\rangle \times \left\langle \cos t - t\sin t, \sin t + t\cos t, \sqrt{3} \right\rangle \\ &= \left\langle -\sqrt{3}\,t^2 \cos t, -\sqrt{3}\,t^2 \sin t, t^2 \right\rangle \end{aligned}$$

and

$$|\mathbf{r}'(t)| = 2t.$$

Thus

$$\begin{aligned} \kappa &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \\ &= \frac{2\,t^2}{8\,t^3} \\ &= \frac{1}{4t} \end{aligned}$$

Similarly,

$$\begin{aligned} \tau &= \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{|\mathbf{r}'(t) \times \mathbf{r}''(t)|^2} \\ &= \frac{\left\langle -\sqrt{3}\,t^2 \cos t, -\sqrt{3}\,t^2 \sin t, t^2 \right\rangle \cdot \langle -2\sin t - t\cos t, 2\cos t - t\sin t, 0\rangle}{|\left\langle -\sqrt{3}\,t^2 \cos t, -\sqrt{3}\,t^2 \sin t, t^2 \right\rangle|^2} \\ &= \frac{2\sqrt{3}\,t^2 \cos t \sin t + \sqrt{3}\,t^3 \cos^2 t - 2\sqrt{3}\,t^2 \cos t \sin t + \sqrt{3}\,t^3 \sin^2 t + 0}{(2t^2)^2} \\ &= \frac{\sqrt{3}\,t^3}{4t^4} \\ &= \frac{\sqrt{3}}{4t} \end{aligned}$$

Consequently,

$$\tau = \sqrt{3}\,\kappa.$$

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2. [10 marks; avg: 5.8]

2.(a) [5 marks] Find the length of the polar curve with polar equation $r = \theta^2$, for $0 \le \theta \le \sqrt{21}$.

Solution: use the formula for the length of a polar curve.

$$\begin{aligned} L = \int_0^{\sqrt{21}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta &= \int_0^{\sqrt{21}} \sqrt{\theta^4 + 4\,\theta^2}\, d\theta \\ &= \int_0^{\sqrt{21}} \theta\, \sqrt{\theta^2 + 4}\, d\theta \\ (\text{let } u = \theta^2 + 4) &= \frac{1}{2} \int_4^{25} \sqrt{u}\, du \\ &= \left[\frac{u^{3/2}}{3}\right]_4^{25} \\ &= \frac{125}{3} - \frac{8}{3} = 39 \end{aligned}$$

2.(b) [5 marks] Find the interval of convergence of the power series $f(x) = \sum_{k=1}^{\infty} \frac{(x+1)^k}{3^k \sqrt{k}}$.

Solution: the radius of convergence is

$$R = \lim_{k\to\infty} \left| \frac{3^{k+1}\sqrt{k+1}}{3^k \sqrt{k}} \right| = 3 \lim_{k\to\infty} \sqrt{\frac{k+1}{k}} = 3 \cdot 1 = 3.$$

Solution: the open interval of convergence is $(-1-3, -1+3) = (-4, 2)$. Now check to see if the power series converges at the end points:

at $x = -4$, $\sum_{k=1}^{\infty} \frac{(x+1)^k}{3^k \sqrt{k}} = \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$, which converges by the alternating series test;

at $x = 2$, $\sum_{k=1}^{\infty} \frac{(x+1)^k}{3^k \sqrt{k}} = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$, which is a p-series with $p = 1/2 \le 1$, so it diverges.

Answer: the interval of convergence is $[-4, 2)$.

Continued...

3. [20 marks; avg: 9.9] For this question let P be a positive function of t that satisfies the differential equation

$$\frac{dP}{dt} = k\,P\left(1 - \frac{P}{L}\right)$$

for some positive constants k and L.

Average this page: 5.3/10

(a) [5 marks] Without solving the differential equation, find its equilibrium solution, and explain why it is a stable equilibrium.

Solution: k, L and P are all positive, so

$$\frac{dP}{dt} = 0 \Rightarrow 1 - \frac{P}{L} = 0 \Rightarrow P = L.$$

This equilibrium value is stable since

$$P < L \Rightarrow \frac{P}{L} < 1 \Rightarrow 1 - \frac{P}{L} > 0 \Rightarrow \frac{dP}{dt} > 0, \text{ so } P \text{ will increase back towards L;}$$

and

$$P > L \Rightarrow \frac{P}{L} > 1 \Rightarrow 1 - \frac{P}{L} < 0 \Rightarrow \frac{dP}{dt} < 0, \text{ so } P \text{ will decrease back towards L.}$$

(b) [5 marks] For which value of P does P grow fastest?

Solution: rewrite $\frac{dP}{dt}$ as

$$\frac{dP}{dt} = k\,P\left(1 - \frac{P}{L}\right) = k\,P - k\left(\frac{P^2}{L}\right)$$

and differentiate implicitly:

$$\frac{d^2P}{dt^2} = k\frac{dP}{dt} - k\,\frac{2P}{L}\frac{dP}{dt} = k\frac{dP}{dt}\left(1 - \frac{2P}{L}\right).$$

Then

$$\frac{d^2P}{dt^2} = 0 \Rightarrow 1 - \frac{2P}{L} = 0 \Rightarrow P = \frac{L}{2}.$$

So $\frac{dP}{dt}$ is maximized at

$$P = \frac{L}{2}.$$

Aside: this value of P is also the P-coordinate of the inflection point on the graph of P.

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(c) [10 marks; avg: 4.6] Suppose that $k = 0.3$, and that when $t = 0$ the initial value of P is $P_0 = 200$ and its rate of increase is $P_0' = 48$. **Solve** for P as a function of t and then **sketch** the graph of P, indicating inflection points and asymptotes, if any.

Solution: use the initial conditions to find L :

$$\begin{aligned} P_0 = 200,\ P_0' = 48 \quad &\Rightarrow \quad 48 = 0.3 \times 200 \left(1 - \frac{200}{L}\right) \\ &\Rightarrow \quad \frac{48}{60} = 1 - \frac{200}{L} \\ &\Rightarrow \quad \frac{200}{L} = \frac{1}{5} \\ &\Rightarrow \quad \frac{L}{200} = 5 \\ &\Rightarrow \quad L = 1000 \end{aligned}$$

Now the initial value problem is

$$\frac{dP}{dt} = 0.3\,P\left(1 - \frac{P}{1000}\right);\ P_0 = 200.$$

This is a logistic differential equation, for which the solution (if you memorized it) is

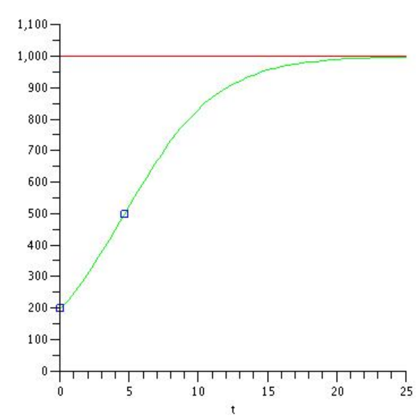
$$P = \frac{1000}{1 + A\,e^{-0.3t}}.$$

To find A let $t = 0$ and $P = 200$:

$$200 = \frac{1000}{1 + A} \Rightarrow A = 4.$$

Or, separate variables:

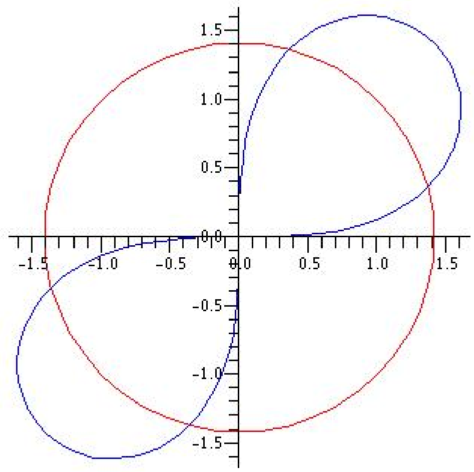
$$\begin{aligned} &\int \frac{1000\,dP}{P(1000 - P)} = \int 0.3\,dt \\ \Rightarrow\ &\int \left(\frac{1}{P} + \frac{1}{1000 - P}\right) dP = 0.3\,t + C \\ \Rightarrow\ &\ln P - \ln(1000 - P) = 0.3\,t + C \\ \Rightarrow\ &\ln \frac{P}{1000 - P} = 0.3\,t + C \\ \Rightarrow\ &\frac{P}{1000 - P} = e^C\,e^{0.3t} \\ \Rightarrow\ &\frac{1000 - P}{P} = A\,e^{-0.3t} \\ \Rightarrow\ &\frac{1000}{P} = 1 + A\,e^{-0.3t} \\ \Rightarrow\ &P = \frac{1000}{1 + A\,e^{-0.3t}} \end{aligned}$$



$$P = \frac{1000}{1 + 4e^{-0.3t}}$$

The graph of P is in green and the horizontal asymptote at $L = 1000$ is in red. Your graph should have an inflection point at $(10/3\ \ln 4, 500) \approx (4.6, 500)$ and the initial value $(0, 200)$.

4. [10 marks; avg: 5.5] Find the area of the region within the curve with polar equation $r^2 = 4\sin(2\theta)$ but outside the circle with polar equation $r = \sqrt{2}$.
(See the curves to the right, which you should label.)



Solution: in the graphs to the right, the circle $r_1 = \sqrt{2}$ is in red, and the curve with equation $r_2^2 = 4\sin(2\theta)$ is in blue. The loop in the first quadrant is $r_2 > 0$; the loop in the third quadrant is $r_2 < 0$.

Intersection points: set $r_1 = r_2$:

$$2 = 4\sin(2\theta) \Rightarrow \sin(2\theta) = \frac{1}{2} \Rightarrow 2\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12} \text{ or } \frac{5\pi}{12}.$$

Calculation of Area: by symmetry,

$$\begin{aligned} A = 2\left(\frac{1}{2}\int_{\pi/12}^{5\pi/12} \left(r_2^2 - r_1^2\right)\, d\theta\right) &= \int_{\pi/12}^{5\pi/12} (4\sin(2\theta) - 2)\, d\theta \\ &= \left[-2\cos(2\theta) - 2\theta\right]_{\pi/12}^{5\pi/12} \\ &= -2\left(-\frac{\sqrt{3}}{2}\right) - \frac{5\pi}{6} + 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6} \\ &= 2\sqrt{3} - \frac{2\pi}{3} \end{aligned}$$

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5. [10 marks; avg: 3.6] For this question i is the 'imaginary' square root of -1. That is, $i = \sqrt{-1}$ and $i^2 = -1$.

(a) [6 marks] Using appropriate power series, derive Euler's formula: $e^{i\,\theta} = \cos\theta + i\,\sin\theta$.

Solution: as done in class. Let $x = i\,\theta$ in the power series of e^x :

$$\begin{aligned} e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} &= 1 + i\theta + i^2\frac{\theta^2}{2!} + i^3\frac{\theta^3}{3!} + i^4\frac{\theta^4}{4!} + i^5\frac{\theta^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\ &= \cos\theta \ + i\sin\theta \end{aligned}$$

(b) [2 marks] What is the value of $e^{i\pi/2}$?

Solution: use Euler's formula.

$$e^{i\pi/2} = \cos(\pi/2) + i\,\sin(\pi/2) = 0 + i = i.$$

(c) [2 marks] What is the value of i^i ?

Solution: the interesting thing here is that i^i turns out to be real:

$$\begin{aligned} i = e^{i\pi/2} \quad &\Rightarrow \quad i^i = \left(e^{i\pi/2}\right)^i \\ &\Rightarrow \quad i^i = e^{i^2\pi/2} \\ &\Rightarrow \quad i^i = e^{-\pi/2} \approx 0.21 \end{aligned}$$

6. [20 marks; avg: 12.2] A 1-kg block hangs on a spring with constant $k = 100$ N/m. Let y be the displacement from equilibrium of the spring at time t, measured in seconds. The block is lifted 12 m above its equilibrium position and let go.

(a) [8 marks; avg: 5.0] Solve the corresponding initial value problem

$$\frac{d^2y}{dt^2} + 100\,y = 0;\ \ y_0 = 12, y_0' = 0$$

for y as a function of t and graph the solution, for $0 \le t \le \pi$, indicating its amplitude and period.

Solution: find the roots of the auxiliary quadratic.

$$r^2 + 100 = 0 \Rightarrow r^2 = -100 \Rightarrow r = \pm 10i,$$

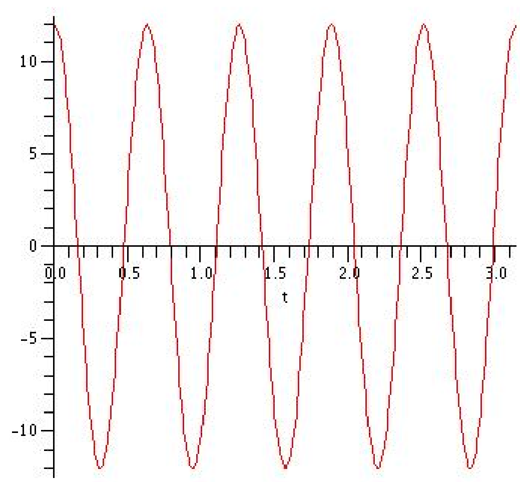
so

$$y = A\cos(10t) + B\sin(10t) \text{ and } y' = -10A\sin(10t) + 10B\cos(10t).$$

But $y_0 = 12 \Rightarrow A = 12$, and $y_0' = 0 \Rightarrow B = 0$, thus

$$y = 12\cos(10t),$$

which represents a sinusoidal wave with amplitude 12 and period $\pi/5$. The graph is below:



Your graph should show five complete cycles, have amplitude 12, and pass through $(0, 12)$.

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(b) [12 marks; avg: 7.1] Now suppose the block-spring system is in a surrounding medium with a damping coefficient of $c = 12$ kg/s. Solve the corresponding initial value problem

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 100\,y = 0;\ \ y_0 = 12, y_0' = 0$$

and express your answer as a product of three terms: a constant, an exponential function, and a single trigonometric function—sine or cosine.

Solution: now

$$r^2 + 12r + 100 = 0 \Rightarrow r = \frac{-12 \pm \sqrt{144 - 400}}{2} = -6 \pm 8\ i,$$

so

$$y = Ae^{-6t}\cos(8t) + Be^{-6t}\sin(8t)$$

and

$$y' = -6Ae^{-6t}\cos(8t) - 8Ae^{-6t}\sin(8t) - 6Be^{-6t}\sin(8t) + 8Be^{-6t}\cos(8t).$$

Use the initial conditions to find that $A = 12$ and

$$-6A + 8B = 0 \Leftrightarrow B = 9.$$

Then

$$\begin{aligned} y &= 12\,e^{-6t}\cos(8t) + 9\,e^{-6t}\sin(8t) \\ &= 15\,e^{-6t}\left(\frac{12}{15}\cos(8t) + \frac{9}{15}\sin(8t)\right) \\ &= 15\,e^{-6t}\sin\left(8t + \alpha\right),\ \text{with}\ \sin\alpha = \frac{4}{5},\ \cos\alpha = \frac{3}{5},\ \alpha = \tan^{-1}\frac{4}{3}, \\ \text{or}\ \ y &= 15\,e^{-6t}\cos\left(8t - \beta\right),\ \text{with}\ \cos\beta = \frac{4}{5},\ \sin\beta = \frac{3}{5},\ \beta = \tan^{-1}\frac{3}{4}. \end{aligned}$$

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