University of Toronto
Faculty of Applied Science and Engineering
Solutions to Final Examination, June 2016
Duration: 2 and 1/2 hrs
First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

MAT187H1F - Calculus II

Examiner: D. Burbulla

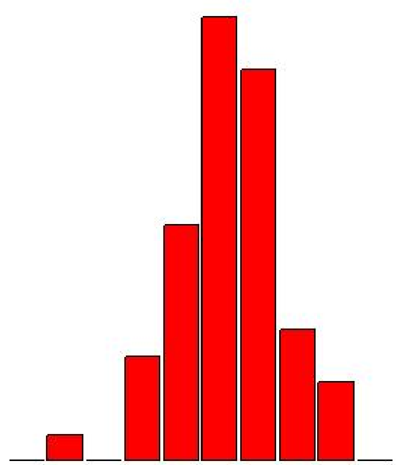
Exam Type: A. Aids permitted: Formula Sheet, and Casio FX-991 or Sharp EL-520 calculator.

General Comments:

1. Considering that more than 80% of this exam was completely computational, the results on this exam were surprisingly low. Eg. 23 of 54 students could not solve the quadratic equation in 6(c) correctly! And many students used incorrect formulas for area and length, even though they were supplied!
2. Interestingly, the only question with a failing average, Question 3, was based on a WeBWorK homework problem. Also: in this question, very few students knew what 'stable equilibrium' meant.
3. Every page was done perfectly at least once, except for pages 3, 4 and 9.

Breakdown of Results: 54 students wrote this exam. The marks ranged from 14% to 89%, and the average was 56.61%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0%
A	5.6%	80-89%	5.6%
B	9.2%	70-79%	9.2%
C	27.8%	60-69%	27.8%
D	31.5%	50-59%	31.5%
F	25.9%	40-49%	16.7%
		30-39%	7.4%
		20-29%	0.0%
		10-19%	1.8%
		0-9%	0.0%



1. [10 marks: avg: 7.15]

The position of a particle at time t is given by $\mathbf{r}(t) = \langle \cos t, \sin t, \ln(\sec t) \rangle$, for $0 \le t < \pi/2$.

(a) [6 marks] Find the velocity, speed and acceleration of the particle at time t.

Solution: let $\mathbf{v}(t)$ be the velocity of the particle at time t. NB. $\sec t > 0$ for $0 \le t < \pi/2$.

$$\mathbf{v}(t) = \frac{d\,\mathbf{r}(t)}{dt} = \left\langle -\sin t, \cos t, \frac{\sec t \tan t}{\sec t} \right\rangle = \langle -\sin t, \cos t, \tan t \rangle;$$

$$\text{speed} = |\mathbf{v}(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + \tan^2 t} = \sqrt{1 + \tan^2 t} = \sec t;$$

$$\text{acceleration} = \frac{d\,\mathbf{v}(t)}{dt} = \langle -\cos t, -\sin t, \sec^2 t \rangle.$$

(b) [4 marks] What is the total distance travelled by the particle for $0 \le t \le \pi/4$?

Solution: distance travelled is the integral of speed.

$$\int_0^{\pi/4} |\mathbf{v}(t)|\, dt = \int_0^{\pi/4} \sec t\, dt = [\ln(\sec t + \tan t)]_0^{\pi/4} = \ln(\sqrt{2} + 1).$$

2. [10 marks: avg: 5.02]

2.(a) [4 marks] Find the length of the logarithmic spiral with polar equation $r = e^{2\theta}$, for $0 \le \theta \le \pi$.

Solution:

$$L = \int_0^\pi \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^\pi \sqrt{e^{4\theta} + 4\, e^{4\theta}}\, d\theta = \sqrt{5} \int_0^\pi e^{2\theta}\, d\theta = \sqrt{5} \left[\frac{e^{2\theta}}{2}\right]_0^\pi = \frac{\sqrt{5}}{2}(e^{2\pi} - 1)$$

2.(b) [6 marks; 2 for each part.] Consider the power series $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{(x-2)^k}{2^k\sqrt{k+1}}$.

(*i*) What is the radius of convergence of $f(x)$?

Solution: the radius of convergence is

$$R = \lim_{k\to\infty} \left| \frac{2^{k+1}\sqrt{k+2}}{2^k\sqrt{k+1}} \right| = 2 \lim_{k\to\infty} \sqrt{\frac{k+2}{k+1}} = 2 \cdot 1 = 2.$$

(*ii*) What is the interval of convergence of $f(x)$?

Solution: the open interval of convergence is $(2-2, 2+2) = (0,4)$. Now check to see if the power series converges at the end points:

$$\text{at } x = 0,\ \sum_{k=0}^{\infty} \frac{(x-2)^k}{2^k\sqrt{k+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{\sqrt{k+1}},\ \text{which converges by the alternating series test;}$$

$$\text{at } x = 4,\ \sum_{k=0}^{\infty} \frac{(x-2)^k}{2^k\sqrt{k+1}} = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k+1}},\ \text{which diverges by the integral test.}$$

Answer: the interval of convergence is $[0,4)$.

(*iii*) What is the value of $f^{(8)}(2)$?

Solution: we have

$$\frac{f^{(8)}(2)}{8!} = \frac{1}{2^8\sqrt{8+1}} \Leftrightarrow f^{(8)}(2) = \frac{8!}{3 \cdot 2^8} = \frac{105}{2} = 52.5$$

3. [20 marks; avg: 6.89] For this question let P be a positive function of t that satisfies the differential equation

$$\frac{dP}{dt} = c\,P\,\ln\frac{6000}{P},$$

for some positive constant c.

(a) [5 marks] Without solving the differential equation, find its equilibrium solution, and explain why it is a stable equilibrium.

Solution: c and P are both positive, so

$$\frac{dP}{dt} = 0 \Rightarrow \ln\frac{6000}{P} = 0 \Rightarrow \frac{6000}{P} = 1 \Rightarrow P = 6000.$$

This equilibrium value is stable since

$$P < 6000 \Rightarrow \frac{6000}{P} > 1 \Rightarrow \ln\frac{6000}{P} > 0 \Rightarrow \frac{dP}{dt} > 0, \text{ so } P \text{ will increase back toward 6000;}$$

and

$$P > 6000 \Rightarrow \frac{6000}{P} < 1 \Rightarrow \ln\frac{6000}{P} < 0 \Rightarrow \frac{dP}{dt} < 0, \text{ so } P \text{ will decrease back toward 6000.}$$

(b) [5 marks] For which value of P does P grow fastest?

Solution: differentiate implicitly and use the product rule. Note that for some calculations it is more convenient to use

$$\ln\frac{6000}{P} = \ln 6000 - \ln P.$$

Then

$$\frac{d^2P}{dt^2} = c\frac{dP}{dt}\ln\frac{6000}{P} + cP\left(-\frac{1}{P}\right)\frac{dP}{dt} = c\frac{dP}{dt}\left(\ln\frac{6000}{P} - 1\right);$$

$$\frac{d^2P}{dt^2} = 0 \Rightarrow \ln\frac{6000}{P} = 1 \Rightarrow \frac{6000}{P} = e \Rightarrow P = \frac{6000}{e}.$$

So $\frac{dP}{dt}$ is maximized at

$$P = \frac{6000}{e}.$$

Aside: this value of P is also the P-coordinate of the inflection point on the graph of P.

(c) [10 marks] Given that $c = 0.2$ and $0 < P < 6000$, **solve** for P as a function of t if $P = 300$ when $t = 0$; and then **sketch** the graph of P, indicating inflection points and asymptotes, if any.

Solution: separate variables.

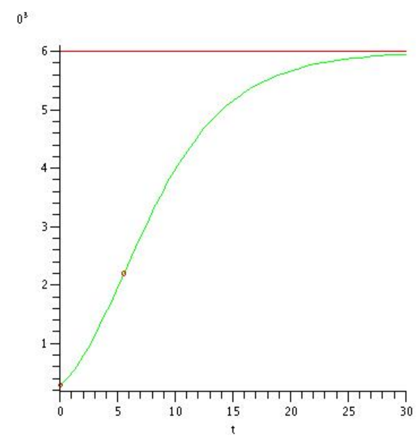
$$\begin{aligned}
\frac{dP}{dt} = \frac{P}{5} \ln \frac{6000}{P} \quad &\Rightarrow \quad \int \frac{dP}{P(\ln 6000 - \ln P)} = \frac{1}{5} \int dt \\
(\text{let } u = \ln 6000 - \ln P) \quad &\Rightarrow \quad -\int \frac{du}{u} = \frac{t}{5} + C \\
&\Rightarrow \quad -\ln u = \frac{t}{5} + C, \text{ since } u > 0, \\
&\Rightarrow \quad -\ln(\ln 6000 - \ln P) = \frac{t}{5} + C
\end{aligned}$$

To find C let $t = 0$ and $P = 300$:

$$C = -\ln(\ln 6000 - \ln 300) = -\ln(\ln 20).$$

Then

$$\begin{aligned}
& -\ln(\ln 6000 - \ln P) = \frac{t}{5} - \ln(\ln 20) \\
\Rightarrow \quad & \ln(\ln 6000 - \ln P) = -\frac{t}{5} + \ln(\ln 20) \\
\Rightarrow \quad & \ln\left(\ln \frac{6000}{P}\right) = -\frac{t}{5} + \ln(\ln 20) \\
\Rightarrow \quad & \ln \frac{6000}{P} = e^{-t/5} \ln 20 = \ln 20^{(e^{-t/5})} \\
\Rightarrow \quad & \frac{6000}{P} = 20^{(e^{-t/5})} \\
\Rightarrow \quad & P = \underbrace{\frac{6000}{20^{(e^{-t/5})}} = 300 \cdot 20^{1-e^{-t/5}}}_{\text{either one will do}}
\end{aligned}$$



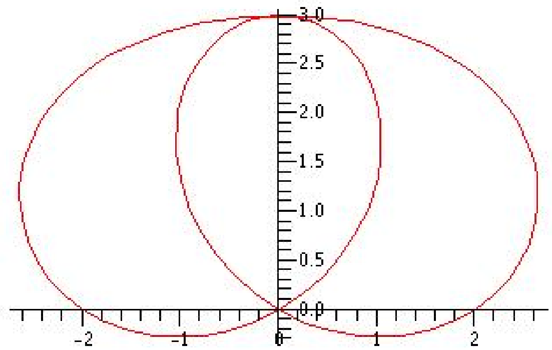
Your graph should have a horizontal asymptote at $P = 6000$, an inflection point at $(5\ln(\ln 20), 6000/e)$ and an initial value $(0, 300)$. NB: the vertical scale on the graph above is in 1000's.

4. [20 marks; avg: 12.2] Let $f(\theta) = 3\sin\theta + 2\cos^2\theta$.

(a) [10 marks] Fill out the short table of values below, and then plot the polar graph of $r = f(\theta)$.

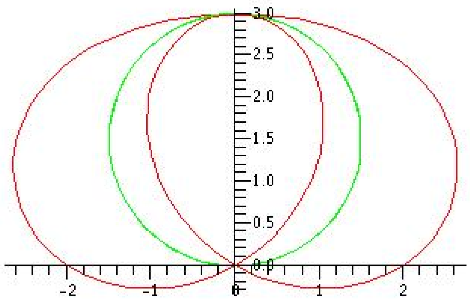
Solution:

θ	$r = f(\theta)$	(x, y)
0	2	$(2, 0)$
$\pi/2$	3	$(0, 3)$
π	2	$(-2, 0)$
$7\pi/6$	0	$(0, 0)$
$3\pi/2$	-3	$(0, 3)$
$11\pi/6$	0	$(0, 0)$
2π	2	$(2, 0)$



(b) [10 marks] Find the area of the region within the polar curve with equation $r = f(\theta)$ but outside the circle with polar equation $r = 3\sin\theta$. (4 marks for setting things up; 6 for your calculations.)

Solution: this is not a one-step problem. You need at least three calculations to get the area.



The graphs of $r = f(\theta)$, red, and the circle $r = 3\sin\theta$, green, are to the left. We need the area outside the green curve but inside the red curve. The easiest way is to calculate all the area within the red curve from $\theta = -\pi/6$ to $\theta = \pi/2$, double it, and then subtract the area within the green circle. (The inner red loop is irrelevant, since it is entirely within the green circle.)

Setting things up:

$$A = 2\left(\frac{1}{2}\int_{-\pi/6}^{\pi/2}(f(\theta))^2\,d\theta\right) - \left(\frac{3}{2}\right)^2\pi \;=\; \int_{-\pi/6}^{\pi/2}(3\sin\theta + 2\cos^2\theta)^2\,d\theta - \frac{9\pi}{4}$$

$$= \int_{-\pi/6}^{\pi/2}(9\sin^2\theta + 12\cos^2\theta\,\sin\theta + 4\cos^4\theta)\,d\theta - \frac{9\pi}{4}$$

Calculations: use double angle and reduction formulas, and $u = \cos\theta$ substitution, as needed.

$$\int_{-\pi/6}^{\pi/2} 9\sin^2\theta\,d\theta = \frac{9}{2}\int_{-\pi/6}^{\pi/2}(1-\cos(2\theta))\,d\theta = \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_{-\pi/6}^{\pi/2} = \frac{9\pi}{4} + \frac{3\pi}{4} - \frac{9\sqrt{3}}{8} = 3\pi - \frac{9\sqrt{3}}{8};$$

$$\int_{-\pi/6}^{\pi/2} 12\cos^2\theta\,\sin\theta\,d\theta = -12\int_{\sqrt{3}/2}^{0} u^2\,du = 12\int_{0}^{\sqrt{3}/2} u^2\,du = 4\left[u^3\right]_0^{\sqrt{3}/2} = \frac{3\sqrt{3}}{2};$$

$$\int_{-\pi/6}^{\pi/2} 4\cos^4\theta\,d\theta = \left[\cos^3\theta\,\sin\theta\right]_{-\pi/6}^{\pi/2} + 3\int_{-\pi/6}^{\pi/2}\cos^2\theta\,d\theta = \frac{3\sqrt{3}}{16} + \frac{3}{2}\int_{-\pi/6}^{\pi/2}(1+\cos(2\theta))\,d\theta$$

$$= \frac{3\sqrt{3}}{16} + \frac{3}{2}\left[\theta + \frac{\sin(2\theta)}{2}\right]_{-\pi/6}^{\pi/2} = \frac{3\sqrt{3}}{16} + \frac{3\pi}{4} + \frac{\pi}{4} + \frac{3\sqrt{3}}{8} = \frac{9\sqrt{3}}{16} + \pi.$$

So, at last,

$$A = \left(3\pi - \frac{9\sqrt{3}}{8} + \frac{3\sqrt{3}}{2} + \frac{9\sqrt{3}}{16} + \pi\right) - \frac{9\pi}{4} = \frac{15\sqrt{3}}{16} + \frac{7\pi}{4}.$$

5. [20 marks; avg: 13.76] Consider the curve in space with vector equation $\mathbf{r}(t) = \langle 13\sin(2t), 12\cos(2t), 5\cos(2t)\rangle$.

(a) [10 marks] Find the curvature, κ, of the curve.

Solution: one way is first to find the unit tangent vector, $\mathbf{T}(t)$.

$$\begin{aligned}
\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} &= \frac{\langle 26\cos(2t), -24\sin(2t), -10\sin(2t)\rangle}{|\langle 26\cos(2t), -24\sin(2t), -10\sin(2t)\rangle|} \\
&= \frac{\langle 26\cos(2t), -24\sin(2t), -10\sin(2t)\rangle}{\sqrt{26^2\cos^2(2t) + 24^2\sin^2(2t) + 10^2\sin^2(2t)}} \\
&= \frac{\langle 26\cos(2t), -24\sin(2t), -10\sin(2t)\rangle}{\sqrt{26^2\cos^2(2t) + 26^2\sin^2(2t)}} \\
&= \frac{1}{26}\langle 26\cos(2t), -24\sin(2t), -10\sin(2t)\rangle \\
&= \left\langle \cos(2t), -\frac{12}{13}\sin(2t), -\frac{5}{13}\sin(2t)\right\rangle \\
\Rightarrow \kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} &= \frac{1}{26}\left|\left\langle -2\sin(2t), -\frac{24}{13}\cos(2t), -\frac{10}{13}\cos(2t)\right\rangle\right| \\
&= \frac{1}{26}\sqrt{4\sin^2(2t) + \left(\frac{24}{13}\right)^2\cos^2(2t) + \left(\frac{10}{13}\right)^2\cos^2(2t)} \\
&= \frac{1}{26}\sqrt{4\sin^2(2t) + 4\cos^2(2t)} = \frac{2}{26} = \frac{1}{13}
\end{aligned}$$

Alternate Solution: first find an arc length parametrization.

$$s = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 26\,du = 26t \Rightarrow t = \frac{s}{26}$$

$$\Rightarrow \mathbf{r}(s) = \left\langle 13\sin\left(\frac{s}{13}\right), 12\cos\left(\frac{s}{13}\right), 5\cos\left(\frac{s}{13}\right)\right\rangle$$

$$\Rightarrow \mathbf{T}(s) = \mathbf{r}'(s) = \left\langle \cos\left(\frac{s}{13}\right), -\frac{12}{13}\sin\left(\frac{s}{13}\right), -\frac{5}{13}\sin\left(\frac{s}{13}\right)\right\rangle$$

$$\Rightarrow \frac{d\mathbf{T}}{ds} = \left\langle -\frac{1}{13}\sin\left(\frac{s}{13}\right), -\frac{12}{13^2}\cos\left(\frac{s}{13}\right), -\frac{5}{13^2}\cos\left(\frac{s}{13}\right)\right\rangle$$

$$\Rightarrow \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \sqrt{\frac{1}{13^2}\sin^2\left(\frac{s}{13}\right) + \frac{12^2}{13^4}\cos^2\left(\frac{s}{13}\right) + \frac{5^2}{13^4}\cos^2\left(\frac{s}{13}\right)} = \cdots = \frac{1}{13}$$

Or use:

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\langle 26\cos(2t), -24\sin(2t), -10\sin(2t)\rangle \times \langle -52\sin(2t), -48\cos(2t), -20\cos(2t)\rangle|}{|\langle 26\cos(2t), -24\sin(2t), -10\sin(2t)\rangle|^3}$$

$$= \frac{|\langle 0, 520, -1248\rangle|}{26^3} = \frac{|\langle 0, 5, -12\rangle|}{13^2} = \frac{13}{13^2} = \frac{1}{13}$$

(b) [6 marks] Find $\mathbf{N}(t)$ and $\mathbf{B}(t)$, the **unit** normal and binormal vectors, respectively, of the curve.

Solution: let $\mathbf{N}(t)$ and $\mathbf{B}(t)$ be the unit normal and binormal vectors, respectively. Then

$$\begin{aligned}\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} &= \frac{\left\langle -2\sin(2t), -\frac{24}{13}\cos(2t), -\frac{10}{13}\cos(2t)\right\rangle}{\left|\left\langle -2\sin(2t), -\frac{24}{13}\cos(2t), -\frac{10}{13}\cos(2t)\right\rangle\right|} \\ &= \frac{1}{2}\left\langle -2\sin(2t), -\frac{24}{13}\cos(2t), -\frac{10}{13}\cos(2t)\right\rangle \\ &= \left\langle -\sin(2t), -\frac{12}{13}\cos(2t), -\frac{5}{13}\cos(2t)\right\rangle\end{aligned}$$

$$\begin{aligned}\mathbf{B}(t) &= \mathbf{T}(t) \times \mathbf{N}(t) \\ &= \left\langle \cos(2t), -\frac{12}{13}\sin(2t), -\frac{5}{13}\sin(2t)\right\rangle \times \left\langle -\sin(2t), -\frac{12}{13}\cos(2t), -\frac{5}{13}\cos(2t)\right\rangle \\ &= \left\langle \frac{60}{169}\sin 2t\cos 2t - \frac{60}{169}\sin 2t\cos 2t, \frac{5}{13}\sin^2 2t + \frac{5}{13}\cos^2 2t, -\frac{12}{13}\cos^2 2t - \frac{12}{13}\sin^2 2t\right\rangle \\ &= \left\langle 0, \frac{5}{13}, -\frac{12}{13}\right\rangle\end{aligned}$$

(c) [4 marks] Describe the curve. The more specific you can be the better.

Solution: the curve is

- in the plane with equation $5y = 12z$, since $\mathbf{B}(t)$ is constant,
- with constant curvature $\kappa = 1/13$; so it is probably
- a circle
- with radius $r = 13$.

This can be confirmed:

$$|\mathbf{r}(t)| = \sqrt{13^2\sin^2(2t) + 12^2\cos^2(2t) + 5^2\cos^2(2t)} = \sqrt{13^2\sin^2(2t) + 13^2\cos^2(2t)} = 13.$$

So the curve is indeed a circle with centre $(0,0,0)$, radius 13, in the plane with equation $5y = 12z$.

6. [20 marks; avg: 11.59] A 5-kg block hangs on a spring with constant $k = 20$ N/m. Let y be the displacement from equilibrium of the spring at time t, measured in seconds. The block is lifted 4 m above its equilibrium position and let go.

(a) [4 marks] Assuming no resistance or external forcing, explain *briefly* why the the motion of the block is described by the initial value problem: $5\dfrac{d^2y}{dt^2} + 20\,y = 0;\ y_0 = 4, y_0' = 0.$

Solution: let y be the position of the block at time t, where $y = 0$ is the equilibrium position of the spring. We have

$$F = -ky \Rightarrow ma = -ky \Rightarrow 5\frac{d^2y}{dt^2} = -20y \Rightarrow 5\frac{d^2y}{dt^2} + 20\,y = 0;$$

and the initial conditions are $y_0 = 4, y_0' = 0$, since the block is just let go—no initial velocity.

(b) [6 marks] Solve the initial value problem from part (a) and graph the solution, for $0 \le t \le 2\pi$, indicating its amplitude and period.

Solution:

$$5r^2 + 20 = 0 \Rightarrow r^2 = -4 \Rightarrow r = \pm 2i,$$

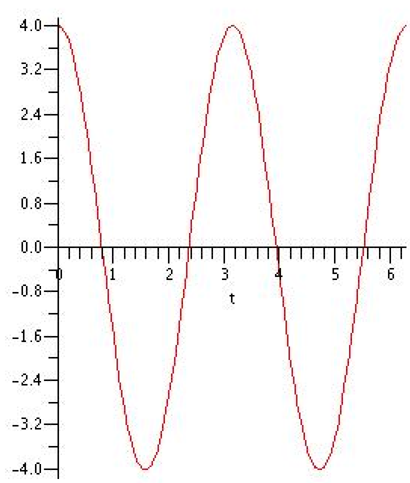
so

$$y = A\cos(2t) + B\sin(2t) \text{ and } y' = -2A\sin(2t) + 2B\cos(2t).$$

But $y_0 = 4 \Rightarrow A = 4$, and $y_0' = 0 \Rightarrow B = 0$, thus

$$y = 4\cos(2t),$$

which represents a sinusoidal wave with amplitude 4 and period π. The graph is below:



(c) [10 marks] Now suppose the block-spring system is in a surrounding medium with a damping coefficient of $c = 12$ kg/s. Solve the corresponding initial value problem

$$5\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 20\,y = 0;\;\; y_0 = 4, y_0' = 0$$

and express your answer as a product of three terms: a constant, an exponential function, and a single trigonometric function—sine or cosine.

Solution: now

$$5r^2 + 12r + 20 = 0 \Rightarrow r = \frac{-12 \pm \sqrt{144 - 400}}{10} = -\frac{6}{5} \pm \frac{8}{5}i,$$

so

$$y = Ae^{-6t/5}\cos\left(\frac{8t}{5}\right) + Be^{-6t/5}\sin\left(\frac{8t}{5}\right)$$

and

$$y' = -\frac{6A}{5}e^{-6t/5}\cos\left(\frac{8t}{5}\right) - \frac{8A}{5}e^{-6t/5}\sin\left(\frac{8t}{5}\right) - \frac{6B}{5}e^{-6t/5}\sin\left(\frac{8t}{5}\right) + \frac{8B}{5}e^{-6t/5}\cos\left(\frac{8t}{5}\right).$$

Use the initial conditions to find that $A = 4$ and

$$-\frac{6A}{5} + \frac{8B}{5} = 0 \Leftrightarrow B = 3.$$

Then

$$\begin{aligned}
y &= 4\,e^{-6t/5}\cos\left(\frac{8t}{5}\right) + 3\,e^{-6t/5}\sin\left(\frac{8t}{5}\right) \\
&= 5\,e^{-6t/5}\left(\frac{4}{5}\cos\left(\frac{8t}{5}\right) + \frac{3}{5}\sin\left(\frac{8t}{5}\right)\right) \\
&= 5\,e^{-6t/5}\sin\left(\frac{8t}{5} + \alpha\right), \text{ with } \sin\alpha = \frac{4}{5},\ \cos\alpha = \frac{3}{5},\ \alpha = \tan^{-1}\frac{4}{3}, \\
\text{or } y &= 5\,e^{-6t/5}\cos\left(\frac{8t}{5} - \beta\right), \text{ with } \cos\beta = \frac{4}{5},\ \sin\beta = \frac{3}{5},\ \beta = \tan^{-1}\frac{3}{4}.
\end{aligned}$$

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