University of Toronto

Faculty of Applied Science and Engineering

Final Examination, June 2015

Duration: 2 and 1/2 hrs

First Year - CHE, CIV, CPE, ELE, ENG, IND, LME, MEC, MMS

Solutions to MAT187H1F - Calculus II

Examiner: D. Burbulla

This exam consists of 8 questions. Each question is worth 10 marks. **Total Marks: 80**

General Comments:

1. In Question 8, about ten students thought $\sqrt{a^2+b^2} = a+b$. And in Question 7(b), very few students included the absolute value sign in the length of $\mathbf{T}'(t)$. Some students thought $1/(a+b) = 1/a + 1/b$.

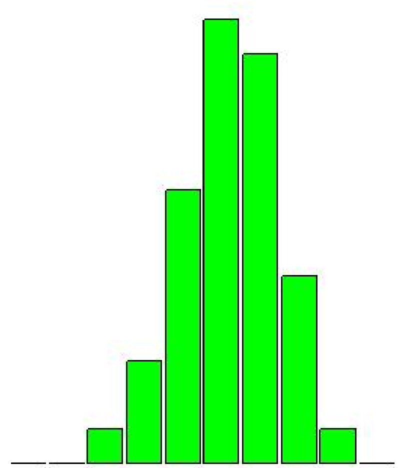
2. Questions 2, 7 and 8 had a top mark of only 9 out of 10; all the other questions were done perfectly at least once. Generally speaking, the calculations in Question 7 and 8 were done *very* poorly! They are not as difficult—or as long—as some of you made them out to be.

3. Considering I told you there would be a logistic growth question on the exam, the results on Question 1 were very disappointing.

4. In Question 7(b) many students forgot the chain rule, or thought $\cos u^2 = \cos^2 u$.

Breakdown of Results: 87 students wrote this exam. The marks ranged from 26.25% to 83.75%, and the average was 56.44%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right.

Grade	%	Decade	%
		90-100%	0.0%
A	2.3%	80-89%	2.3%
B	12.6 %	70-79%	12.6%
C	27.6%	60-69%	27.6%
D	29.9%	50-59%	29.9%
F	27.6%	40-49%	18.4%
		30-39%	6.9%
		20-29%	2.3%
		10-19%	0.0%
		0-9%	0.0%



PART I : Answer all of the following short-answer questions by putting your answers in the blanks. No explanation is necessary, but part marks are possible if you show your work and your answer is wrong.

1. [10 marks; avg: 5.72] Consider the initial value problem for $t \geq 0$.

$$\text{DE: } \frac{dP}{dt} = 0.1\,P\left(1 - \frac{P}{400}\right), \quad \text{IC: } P_0 = 50.$$

(a) [2 marks] The stable equilibrium solution to the DE is $P =$ 400

(b) [2 marks] $\lim_{t\to\infty} P =$ 400

(same as part (a))

(c) [2 marks] The solution to the initial value problem is $P =$ $\dfrac{400}{1+7e^{-t/10}}$

$$P = \frac{400}{1+Ae^{-t/10}};\; P_0 = \frac{400}{1+Ae^{0}} \Leftrightarrow 1 + A = 8 \Leftrightarrow A = 7.$$

(d) [2 marks] The graph of P has one inflection point at $(t, P) =$ $(10\ln 7, 200)$

$$200 = \frac{400}{1+7e^{-t/10}} \Leftrightarrow 1 + 7e^{-t/10} = 2 \Leftrightarrow 7 = e^{t/10}.$$

(e) [2 marks] How long will it take until $P = 100$? $t \approx$ 8.473

$$100 = \frac{400}{1+7e^{-t/10}} \Leftrightarrow 1 + 7e^{-t/10} = 4 \Leftrightarrow \frac{7}{3} = e^{t/10} \Leftrightarrow \frac{t}{10} = \ln\left(\frac{7}{3}\right).$$

2. [10 marks; avg: 5.09] Let $f(x) = \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{x^{k+2}}{2^k} \right)$

(a) [2 marks] The first four terms of this series are $\underline{\quad x^2 + \frac{x^3}{4} + \frac{x^4}{12} + \frac{x^5}{32} \quad}$

(b) [2 marks] The radius of convergence of f is $R = \underline{\quad 2 \quad}$

(c) [2 marks] The interval of convergence of f is $\underline{\quad [-2, 2) \quad}$

(d) [4 marks] $f(x) = \underline{\quad -2x \ln\left(1 - \frac{x}{2}\right) \quad}$ (not as a power series)

$$
\begin{aligned}
f(x) &= x \sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{x^{k+1}}{2^k} \right) \\
&= x \sum_{k=0}^{\infty} \int \frac{x^k}{2^k} \, dx \\
&= x \int \left(\sum_{k=0}^{\infty} \frac{x^k}{2^k} \right) dx \\
&= x \int \frac{1}{1 - x/2} \, dx \\
&= x \left(-2 \ln(1 - x/2) \right) \\
&= -2x \ln(1 - x/2)
\end{aligned}
$$

Or use the series for $\ln(1 - x)$ supplied on the formula sheet, but it's not completely obvious:

$$
\begin{aligned}
\sum_{k=0}^{\infty} \frac{1}{k+1} \left(\frac{x^{k+2}}{2^k} \right) &= x^2 + \frac{x^3}{4} + \frac{x^4}{12} + \frac{x^5}{32} + \cdots \\
&= x \left(x + \frac{x^2}{4} + \frac{x^3}{12} + \frac{x^4}{32} + \cdots \right) \\
&= 2x \left(\frac{x}{2} + \frac{1}{2} \frac{x^2}{4} + \frac{1}{3} \frac{x^3}{8} + \frac{1}{4} \frac{x^4}{2^4} + \cdots \right) \\
&= 2x \left(-\ln\left(1 - \frac{x}{2}\right) \right) \\
&= -2x \ln(1 - x/2)
\end{aligned}
$$

PART II : Present **complete** solutions to the following questions in the space provided.

3. [10 marks; avg: 6.34] Consider the logarithmic spiral with polar equation $r = e^{\theta}$.

   (a) [4 marks] What is the length of the spiral for $0 \le \theta \le \pi$?

   Solution:

$$L = \int_0^{\pi} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{e^{2\theta} + e^{2\theta}}\, d\theta = \int_0^{\pi} \sqrt{2}\, e^{\theta}\, d\theta = \left[\sqrt{2}\, e^{\theta}\right]_0^{\pi} = \sqrt{2}\,(e^{\pi} - 1)$$

   (b) [6 marks] Find all the critical[1] points on the spiral for $0 \le \theta \le 2\pi$.

   Solution: parametric equations of the spiral are $x = r\cos\theta$ and $y = r\sin\theta$. So

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{e^{\theta}\sin\theta + e^{\theta}\cos\theta}{e^{\theta}\cos\theta - e^{\theta}\sin\theta} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}.$$

   For horizontal tangents

$$\sin\theta + \cos\theta = 0 \Leftrightarrow \tan\theta = -1 \Leftrightarrow \theta = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}.$$

   For vertical tangents

$$\cos\theta - \sin\theta = 0 \Leftrightarrow \tan\theta = 1 \Leftrightarrow \theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}.$$

   So the four critical points, in Cartesian coordinates, are

$$(x, y) = \left(\frac{e^{\pi/4}}{\sqrt{2}}, \frac{e^{\pi/4}}{\sqrt{2}}\right), \left(-\frac{e^{3\pi/4}}{\sqrt{2}}, \frac{e^{3\pi/4}}{\sqrt{2}}\right), \left(-\frac{e^{5\pi/4}}{\sqrt{2}}, -\frac{e^{5\pi/4}}{\sqrt{2}}\right), \left(\frac{e^{7\pi/4}}{\sqrt{2}}, -\frac{e^{7\pi/4}}{\sqrt{2}}\right).$$

   I will also accept polar coordinates r and θ :

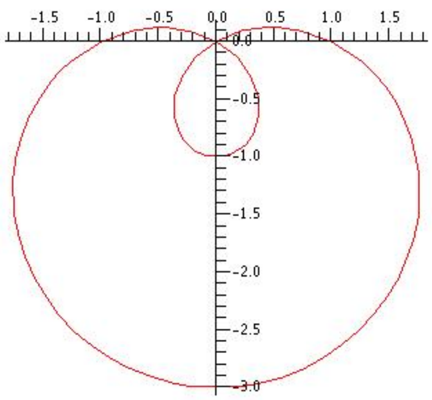
$$(r, \theta) = \left(e^{\pi/4}, \frac{\pi}{4}\right), \left(e^{3\pi/4}, \frac{3\pi}{4}\right), \left(e^{5\pi/4}, \frac{5\pi}{4}\right), \left(e^{7\pi/4}, \frac{7\pi}{4}\right).$$

   But just giving θ is not enough.

[1] A critical point is a point where the slope of the graph is zero or undefined.

4. [10 marks; avg: 5.17] Plot the limaçon with polar equation $r = 1 - 2\sin\theta$ and then find the area of the region that is inside the outer loop but outside the inner loop.

Solution:



$$r = 0 \Rightarrow \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}.$$

For the inner loop, with area A_i,

$$\frac{\pi}{6} \le \theta \le \frac{5\pi}{6};$$

for the outer loop, with area A_o,

$$\frac{5\pi}{6} \le \theta \le \frac{13\pi}{6}.$$

Then

$$A_o = \frac{1}{2}\int_{5\pi/6}^{13\pi/6} (1 - 2\sin\theta)^2\, d\theta \text{ or } A_o = \int_{5\pi/6}^{3\pi/2} (1 - 2\sin\theta)^2\, d\theta, \text{ by symmetry;}$$

and

$$A_i = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (1 - 2\sin\theta)^2\, d\theta \text{ or } A_i = \int_{\pi/6}^{\pi/2} (1 - 2\sin\theta)^2\, d\theta, \text{ by symmetry.}$$

For each integral we can use

$$\int (1 - 2\sin\theta)^2\, d\theta = \int (1 - 4\sin\theta + 4\sin^2\theta)\, d\theta = \int (3 - 4\sin\theta - 2\cos(2\theta))\, d\theta = 3\theta + 4\cos\theta - \sin(2\theta).$$

Thus

$$A_o = \int_{5\pi/6}^{3\pi/2} (1 - 2\sin\theta)^2\, d\theta = [3\theta + 4\cos\theta - \sin(2\theta)]_{5\pi/6}^{3\pi/2} = 2\pi + \frac{3\sqrt{3}}{2};$$

$$A_i = \int_{\pi/6}^{\pi/2} (1 - 2\sin\theta)^2\, d\theta = [3\theta + 4\cos\theta - \sin(2\theta)]_{\pi/6}^{\pi/2} = \pi - \frac{3\sqrt{3}}{2}.$$

So the area of the region that is inside the outer loop but outside the inner loop is

$$A = A_o - A_i = \pi + 3\sqrt{3}.$$

Alternate Solution: you can also calculate the integral for $0 \le \theta \le 2\pi$, and then subtract the area of the inner loop, but you have to subtract the area of the inner loop *twice*.

$$\frac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta)^2\, d\theta - 2\,A_i = 3\pi - 2\left(\pi - \frac{3\sqrt{3}}{2}\right) = \pi + 3\sqrt{3}.$$

5. [10 marks; avg: 6.69] A small rocket is fired from a launch pad 10 m above the ground with an initial velocity of $\mathbf{v}_0 = \langle\, 300, 400, 500\,\rangle$, in m/s. A cross wind blowing to the north produces an acceleration of the rocket of 2.5 m/s^2. Assume the positive x-axis points east, the positive y-axis points north, and the positive z-axis points up. Answer the following questions.[2] (Ignore air resistance; use $g = 9.8$ m/s^2.)

(a) [4 marks] What are the velocity and position vectors for the rocket at time $t \geq 0$?

Solution: take $\mathbf{r}_0 = \langle\, 0, 0, 10\,\rangle$, $\mathbf{a} = \langle\, 0, 2.5, -9.8\,\rangle$. Then

$$\mathbf{v} = \int \mathbf{a}\, dt = \langle 0, 2.5t, -9.8t\,\rangle + \mathbf{v}_0 = \langle\, 300, 400 + 2.5t, 500 - 9.8t\,\rangle;$$

$$\mathbf{r} = \int \mathbf{v}\, dt = \langle 300t, 400t + 1.25t^2, 500t - 4.9t^2\rangle + \mathbf{r}_0 = \langle 300t, 400t + 1.25t^2, 10 + 500t - 4.9t^2\rangle.$$

(b) [4 marks] How long (to the nearest second) is the rocket in the air, and where on the ground (to the nearest meter) does it land?

Solution: we have $x = 300t,\ y = 400t + 1.25t^2,\ z = 10 + 500t - 4.9t^2$. The rocket lands when

$$z = 0 \Rightarrow 4.9t^2 - 500t - 10 = 0 \Rightarrow t = 102.0608124\ldots$$

So the rocket is in the air for 102 seconds and lands at $x = 30,618$, $y = 53,845$, where we rounded off *after* substituting the decimal value of t. (But values of $x = 30,600$ and $y = 53,805$ will also be accepted.)

(c) [2 marks] What is the maximum height (to the nearest meter) of the rocket during its flight?

Solution: find the critical point of z with respect to t. Since $z = 10 + 500t - 4.9t^2$,

$$\frac{dz}{dt} = 500 - 9.8t = 0 \Leftrightarrow t = 51.02040816\ldots.$$

and the maximum height of the rocket is $z = 12,765$ to the nearest meter.

Note: since $z_0 = 10$, this is not a symmetric trip, and so the maximum value of z does *not* occur at half of the time taken in part (b).

[2] This problem was based on a suggested homework problem.

6. [10 marks; avg: 7.8] Consider the curve with vector equation $\mathbf{r}(t) = \langle\, 3\cos t, 3\sin t, -2\,t^{3/2}\,\rangle$, for $t \geq 0$.

(a) [5 marks] Calculate both $\mathbf{r}'(t)$ and $|\mathbf{r}'(t)|$.

Solution:

$$\mathbf{r}'(t) = \langle\, -3\sin t, 3\cos t, -3\sqrt{t}\,\rangle$$

and

$$|\mathbf{r}'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 9t} = 3\sqrt{1+t}.$$

(b) [5 marks] Find an arc length parameterization of the curve.

Solution:

$$s = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 3\sqrt{1+u}\,du = \Big[2(1+u)^{3/2}\Big]_0^t = 2(1+t)^{3/2} - 2 = 2\left((1+t)^{3/2} - 1\right);$$

consequently

$$(1+t)^{3/2} - 1 = \frac{s}{2} \Rightarrow (1+t)^{3/2} = 1 + \frac{s}{2} \Rightarrow 1 + t = \left(1 + \frac{s}{2}\right)^{2/3} \Rightarrow t = \left(1 + \frac{s}{2}\right)^{2/3} - 1;$$

and an arc length parameterization of the curve is

$$\mathbf{r}(s) = \left\langle 3\cos\left(\left(1+\frac{s}{2}\right)^{2/3} - 1\right), 3\sin\left(\left(1+\frac{s}{2}\right)^{2/3} - 1\right), -2\left(\left(1+\frac{s}{2}\right)^{2/3} - 1\right)^{3/2} \right\rangle.$$

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7. [10 marks; avg: 3.62] The Fresnel integrals $C(t) = \int_0^t \cos u^2\, du$, $S(t) = \int_0^t \sin u^2\, du$ arise in the theory of optics.

(a) [4 marks] What are the Maclaurin series, in sigma notation, for $C(t)$ and $S(t)$?

Solution: use the series for sine and cosine, and then integrate.

$$C(t) = \int_0^t \left(\sum_{k=0}^{\infty} (-1)^k \frac{(u^2)^{2k}}{(2k)!} \right) du = \sum_{k=0}^{\infty} (-1)^k \left(\int_0^t \frac{u^{4k}}{(2k)!} du \right) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{4k+1}}{(4k+1)(2k)!}$$

$$S(t) = \int_0^t \left(\sum_{k=0}^{\infty} (-1)^k \frac{(u^2)^{2k+1}}{(2k+1)!} \right) du = \sum_{k=0}^{\infty} (-1)^k \left(\int_0^t \frac{u^{4k+2}}{(2k+1)!} du \right) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{4k+3}}{(4k+3)(2k+1)!}$$

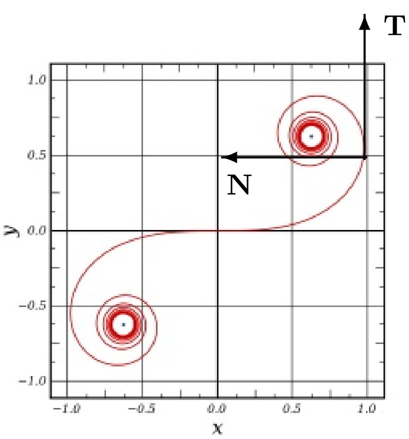
(b) [6 marks] The Euler spiral, shown to the right, is the curve with vector equation $\mathbf{r}(t) = \langle C(t), S(t) \rangle$. For this curve, find

(i) the unit tangent vector $\mathbf{T}(t)$,

(ii) the unit normal vector $\mathbf{N}(t)$,

(iii) and the curvature κ.

Illustrate your results, for $t = \sqrt{\pi/2}$, on the graph to the right. NB: you do not need part (a) to do part (b).



Solution:

(i) $\mathbf{r}'(t) = \langle \cos t^2, \sin t^2 \rangle$, which is already a unit vector. So $\mathbf{T}(t) = \langle \cos t^2, \sin t^2 \rangle$.

(ii) $\mathbf{T}'(t) = \langle -2t \sin t^2, 2t \cos t^2 \rangle$ and $|\mathbf{T}'(t)| = \sqrt{4t^2 \sin^2 t^2 + 4t^2 \cos^2 t^2} = \sqrt{4t^2} = 2|t|$, so

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \begin{cases} \langle -\sin t^2, \cos t^2 \rangle & \text{if } t > 0 \\ \langle \sin t^2, -\cos t^2 \rangle & \text{if } t < 0 \end{cases}$$

(iii)

$$\kappa = \left| \frac{\mathbf{T}'(t)}{\mathbf{r}'(t)} \right| = \frac{2|t|}{1} = 2|t|.$$

At $t = \sqrt{\pi/2}$ there is a vertical tangent line, since $dx/dt = 0$, and

$$\mathbf{T}\left(\sqrt{\frac{\pi}{2}}\right) = \langle 0, 1 \rangle, \ \mathbf{N} = \left(\sqrt{\frac{\pi}{2}}\right) = \langle -1, 0 \rangle, \ \kappa = \sqrt{2\pi}.$$

8. [10 marks; avg: 4.7] Let $\mathbf{r}(t) = \langle\, t, 1-t, t^2 \,\rangle$ be the vector equation of a curve in $\mathbb{R}^3$.

(a) [8 marks] Find $\mathbf{T}(t), \mathbf{N}(t)$ and $\mathbf{B}(t)$, the **unit** tangent, normal and binormal vectors, respectively.

Solution: $\mathbf{r}'(t) = \langle\, 1, -1, 2t \,\rangle$ and $|\mathbf{r}'(t)| = \sqrt{2+4t^2}$, so

$$\mathbf{T}(t) = \left\langle \frac{1}{\sqrt{2+4t^2}}, \frac{-1}{\sqrt{2+4t^2}}, \frac{2t}{\sqrt{2+4t^2}} \right\rangle.$$

Slightly more complicated calculations give

$$\mathbf{T}'(t) = \left\langle \frac{-4t}{(2+4t^2)^{3/2}}, \frac{4t}{(2+4t^2)^{3/2}}, \frac{4}{(2+4t^2)^{3/2}} \right\rangle,\ |\mathbf{T}'(t)| = \frac{4\sqrt{1+2t^2}}{(2+4t^2)^{3/2}},$$

so

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \left\langle \frac{-t}{\sqrt{1+2t^2}}, \frac{t}{\sqrt{1+2t^2}}, \frac{1}{\sqrt{1+2t^2}} \right\rangle.$$

Then

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \left\langle \frac{-1-2t^2}{\sqrt{2}\,(1+2t^2)}, \frac{-1-2t^2}{\sqrt{2}\,(1+2t^2)}, \frac{t-t}{\sqrt{2}\,(1+2t^2)} \right\rangle = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle.$$

(b) [2 marks] Answer **one** and **only** one of the following two questions:

(i) What can you conclude about the osculating planes of the curve?

(ii) What is the maximum curvature of the curve?

Solution (i) : since $\mathbf{B}(t)$ is a constant vector, all the osculating planes of the curve must be parallel. In fact, all osculating planes to the curve are the same, and have equation

$$x + y = 1$$

since the curve $\mathbf{r}(t) = \langle\, t, 1-t, t^2 \,\rangle$ is entirely contained within this plane.

Solution (ii) : the curvature is

$$\kappa = \left| \frac{\mathbf{T}'(t)}{\mathbf{r}'(t)} \right| = \frac{1}{(1+2t^2)^{3/2}},$$

which has maximum value $\kappa = 1$ at $t = 0$.

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