University of Toronto
FACULTY OF APPLIED SCIENCE AND ENGINEERING

Solutions to

**FINAL EXAMINATION, JUNE, 2014**

First Year - CHE, CIV, IND, LME, MEC, MSE

**MAT187H1F - CALCULUS II**

Exam Type: A

**Examiner:** D. Burbulla

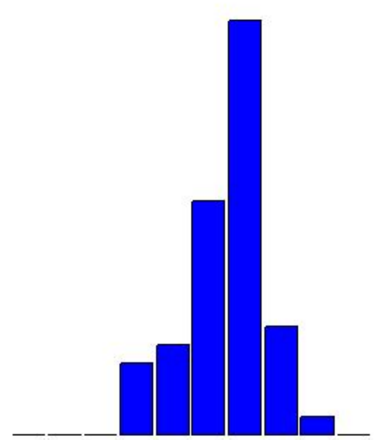
**INSTRUCTIONS:** Attempt all 9 questions; present your solutions in the exam books provided. You must show your work and give complete explanations to get full marks. The value of each question is indicated in parentheses beside the question number. Only aids permitted: Casio FX-991 or Sharp EL-520 calculator. **Total Marks:** 100.

**Comments:**

1. The range on each question was 0 to perfect; Questions 2 and 8 had failing averages.
2. In Question 2 many students were totally befuddled by all the logarithms!
3. Question 8 is actually short and straightforward if you use series.
4. Question 7 and the integral in Question 4 were the only real challenging parts of the exam.

**Breakdown of Results:** 52 students wrote this exam. The marks ranged from 33% to 80%, and the average was 59.7%. Some statistics on grade distributions are in the table on the left, and a histogram of the marks (by decade) is on the right. (For the purposes of calculating the final grades, I added 2 marks to everybody's exam score to raise the average on the exam to 61.7%, the same as last year. The overall average in the course ended up at 66.4%.)

| Grade | % | Decade | % |
|---|---|---|---|
| | | 90-100% | 0.0% |
| A | 1.9% | 80-89% | 1.9% |
| B | 11.6% | 70-79% | 11.6% |
| C | 44.2% | 60-69% | 44.2% |
| D | 25.0% | 50-59% | 25.0% |
| F | 17.3% | 40-49% | 9.6% |
| | | 30-39% | 7.7% |
| | | 20-29% | 0.0% |
| | | 10-19% | 0.0% |
| | | 0-9% | 0.0% |



1. [avg: 10.0/12] Find all the critical points of the function $f(x,y) = 6xy^2 - 2x^3 + 3y^4$, and determine if they are maximum points, minimum points, or saddle points.

**Solution:** Let $z = f(x,y)$. Then

$$\frac{\partial z}{\partial x} = 6y^2 - 6x^2 \text{ and } \frac{\partial z}{\partial y} = 12xy + 12y^3.$$

**Critical points:**

$$\left.\begin{array}{rcl} 6y^2 - 6x^2 & = & 0 \\ 12xy + 12y^3 & = & 0 \end{array}\right\} \Rightarrow \left\{ \begin{array}{c} x^2 = y^2 \\ y = 0 \text{ or } x = -y^2 \end{array}\right.$$

If $y = 0$ then $x = 0$; if $x = -y^2$ then $y^4 = y^2$. So the three critical points are

$$(x,y) = (0,0), (x,y) = (-1,1) \text{ or } (x,y) = (-1,-1).$$

**Second Derivative Test:**

$$\frac{\partial^2 z}{\partial x^2} = -12x, \ \frac{\partial^2 z}{\partial y^2} = 12x + 36y^2, \ \frac{\partial^2 z}{\partial x \partial y} = 12y$$

and

$$\Delta = -12x(12x + 36y^2) - 144y^2.$$

At both $(x,y) = (-1, \pm 1)$,

$$\Delta = 144 > 0 \text{ and } \frac{\partial^2 z}{\partial x^2} = 12 > 0,$$

so $f$ has a **minimum** value (of $z = -1$) at both $(x,y) = (-1, \pm 1)$.

At $(x,y) = (0,0)$,

1. $z = 0$, and
2. $\Delta = 0$, so the second derivative test is inconclusive.

**Bonus:** Consider $z = f(x,0) = -2x^3$:

$$x < 0 \Rightarrow z > 0; \quad x > 0 \Rightarrow z < 0.$$

So $z = 0$ can be neither a maximum nor a minimum value of $f$ at $(x,y) = (0,0)$; thus $f$ has a **saddle point** at $(x,y) = (0,0)$.

2. [avg: 2.7/10] Consider an alternative to the logistic equation called the Gompertz equation, which models popluation growth by the differential equation

$$\frac{dP}{dt} = cP \ln\left(\frac{K}{P}\right),$$

where $P$ is the population at time $t$, $c$ and $K$ are both positive constants, and $P > 0$. Solve for $P$ as a function of $t$ if $P = P_0 < K$ at $t = 0$. What is $\lim_{t\to\infty} P$?

**Solution:** separate variables. Use $\ln\left(\frac{K}{P}\right) = \ln K - \ln P$, to make the integration easier.

$$\int \frac{dP}{P(\ln K - \ln P)} = \int c\, dt \Rightarrow -\ln|\ln K - \ln P| = c\,t + C,$$

where we used the substitution $u = \ln K - \ln P$ to integrate the integral on the left. To find $C$ use the initial condition $t = 0, P = P_0$:

$$C = -\ln|\ln K - \ln P_0| = -\ln(\ln K - \ln P_0),$$

since $P_0 < K$. Then

$$\begin{aligned}
\ln|\ln K - \ln P| = -c\,t + \ln(\ln K - \ln P_0) \;&\Rightarrow\; \left|\ln\frac{K}{P}\right| = e^{-ct}\ln\frac{K}{P_0} \\
&\Rightarrow\; \ln\frac{K}{P} = e^{-ct}\ln\frac{K}{P_0},\; \text{since } P = P_0 < K, \text{ if } t = 0 \\
&\Rightarrow\; \frac{K}{P} = e^{e^{-ct}\ln(K/P_0)} = \left(\frac{K}{P_0}\right)^{e^{-ct}} \\
&\Rightarrow\; \frac{P}{K} = \left(\frac{P_0}{K}\right)^{e^{-ct}} \\
&\Rightarrow\; P = K\left(\frac{P_0}{K}\right)^{e^{-ct}}
\end{aligned}$$

Finally,

$$\lim_{t\to\infty} P = K \lim_{t\to\infty}\left(\frac{P_0}{K}\right)^{e^{-ct}} = K\cdot\left(\frac{P_0}{K}\right)^{0} = K\cdot 1 = K.$$

3. [avg: 10.4/12] Find the general solution for each of the following differential equations:

(a) [7 marks] $\dfrac{dy}{dx} + 2\,y\,\tan x = \tan^3 x$

**Solution:** use the method of the integrating factor.

$$\mu = e^{\int 2\tan x\,dx} = e^{2\ln|\sec x|} = |\sec x|^2 = \sec^2 x.$$

Then:

$$y = \frac{1}{\mu}\int \mu\,\tan^3 x\,dx = \cos^2 x\int \sec^2 x\tan^3 x\,dx.$$

This integral can be evaluated by letting $u = \tan x$. Then $du = \sec^2 x\,dx$ and

$$\int \sec^2 x\tan^3 x\,dx = \int u^3\,du = \frac{u^4}{4} + C = \frac{\tan^4 x}{4} + C.$$

Thus

$$y = \cos^2 x\int \sec^2 x\tan^3 x\,dx = \cos^2 x\left(\frac{\tan^4 x}{4} + C\right) = \frac{1}{4}\sin^2 x\tan^2 x + C\cos^2 x.$$

(b) [5 marks] $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 9y = 0$

**Solution:** solve the auxiliary quadratic equation:

$$r^2 + 6r + 9 = 0 \Rightarrow r = -3,$$

repeated. So

$$y = Ae^{-3x} + B\,x\,e^{-3x}.$$

4. [avg: 5.9/10] Suppose $x = \sqrt{t} + \ln t$ and $y = \sqrt{t} - \ln t$, for $5 \leq t \leq 12$, are the parametric equations of a curve in the $xy$-plane. Find the length of the curve.

**Solution:**

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}} + \frac{1}{t} = \frac{\sqrt{t}+2}{2t}; \quad \frac{dy}{dt} = \frac{1}{2\sqrt{t}} - \frac{1}{t} = \frac{\sqrt{t}-2}{2t}.$$

Then

$$\begin{aligned}
L = \int_5^{12} \sqrt{(x'(t))^2 + (y'(t))^2}\,dt &= \int_5^{12} \sqrt{\left(\frac{\sqrt{t}+2}{2t}\right)^2 + \left(\frac{\sqrt{t}-2}{2t}\right)^2}\,dt \\
&= \int_5^{12} \sqrt{\frac{2t+8}{4t^2}}\,dt \\
&= \frac{1}{\sqrt{2}} \int_5^{12} \frac{\sqrt{t+4}}{t}\,dt \\
(\text{let } t+4 = u^2) \quad &= \frac{1}{\sqrt{2}} \int_3^4 \frac{2u^2}{u^2-4}\,du \\
&= \sqrt{2} \int_3^4 \left(1 + \frac{4}{u^2-4}\right) du \\
&= \sqrt{2} \int_3^4 \left(1 + \frac{1}{u-2} - \frac{1}{u+2}\right) du \\
&= \sqrt{2}\left[u + \ln(u-2) - \ln(u+2)\right]_3^4 \\
&= \sqrt{2}(1 + \ln 5 - \ln 3) \\
&= \sqrt{2} + \sqrt{2}\,\ln\left(\frac{5}{3}\right)
\end{aligned}$$

5. [avg: 7.8/12]

(a) [6 marks] Find the interval of convergence of the power series $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^k}{k\,4^k}$.

**Solution:** we have $a_k = \dfrac{(-1)^{k+1}}{k\,4^k}$, so

$$R = \lim_{k\to\infty}\left|\frac{a_k}{a_{k+1}}\right| = \lim_{k\to\infty}\frac{(k+1)\,4^{k+1}}{k\,4^k} = 4\lim_{k\to\infty}\frac{k+1}{k} = 4\times 1 = 4,$$

and the open interval of convergence is $(-R,R) = (-4,4)$. At the endpoint $x = 4$ the series becomes

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}$$

which converges by the alternating series test, since obviously the sequence $1/k$ decreases to zero as $k\to\infty$. But at $x = -4$ the series becomes

$$-\sum_{k=1}^{\infty}\frac{1}{k}$$

which diverges because it is the negative of the harmonic series. So the interval of convergence is $(-4,4]$.

(b) [6 marks] Use series to find an approximation of $\int_0^{1/2}\frac{x^2\,dx}{(1+x^4)^{3/2}}$ correct to within $10^{-5}$.

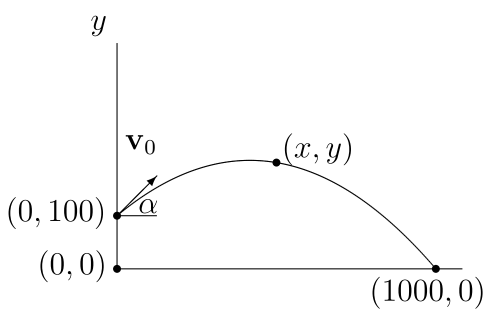
**Solution:** use the binomial series, and integrate term by term:

$$\begin{aligned}
\int_0^{1/2}\frac{x^2\,dx}{(1+x^4)^{3/2}} &= \int_0^{1/2} x^2(1+x^4)^{-3/2}\,dx \\
&= \int_0^{1/2} x^2\left(1-\frac{3x^4}{2}+\frac{15x^8}{8}-\frac{35x^{12}}{16}+\cdots\right)dx \\
&= \int_0^{1/2}\left(x^2-\frac{3x^6}{2}+\frac{15x^{10}}{8}-\frac{35x^{14}}{16}+\cdots\right)dx \\
&= \left[\frac{x^3}{3}-\frac{3x^7}{14}+\frac{15x^{11}}{88}-\frac{7x^{15}}{48}+\cdots\right]_0^{1/2} \\
&= \underbrace{\frac{1}{24}-\frac{3}{1792}+\frac{15}{180224}}-\frac{7}{(48)(2^{15})}+\cdots \\
&\approx 0.040075789,\ \text{correct to within } \frac{7}{(48)(2^{15})}\approx 4.45\times 10^{-6} < 10^{-5},
\end{aligned}$$

by the alternating series test remainder term.

6. [avg: 6.7/10] A projectile is to be fired from the top of a 100-m cliff at a target 1 km from the base of the cliff. The projectile is fired with initial speed $v_0 = 200$ m/sec, at an angle $\alpha$ to the horizontal. Find $\alpha$. (Ignore air resistance; use $g = 9.8$ m/s$^2$.)

**Solution:**




Let the position of the projectile at time $t$ be $(x, y)$ with the position at $t = 0$ set at $(0, 100)$. We have

$$v_0 = 200, x_0 = 0, y_0 = 100$$

and

$$x = 200\cos(\alpha)\,t,\ y = 100+200\sin(\alpha)\,t-4.9\,t^2.$$

The trajectory of the projectile is to pass through the point $(x, y) = (1000, 0)$ whence

$$200\cos(\alpha)\,t = 1000 \text{ and } 100 + 200\sin(\alpha)\,t - 4.9\,t^2 = 0.$$

Substitute $t = 5\sec\alpha$ into the quadratic for $t$ and solve for $\tan\alpha$:

$$\begin{aligned}
& 4.9(5\sec\alpha)^2 - 1000\sin\alpha\sec\alpha - 100 = 0 \\
\Leftrightarrow\ & 122.5\sec^2\alpha - 1000\tan\alpha - 100 = 0 \\
\Leftrightarrow\ & 122.5(1 + \tan^2\alpha) - 1000\tan\alpha - 100 = 0 \\
\Leftrightarrow\ & 122.5\tan^2\alpha - 1000\tan\alpha + 22.5 = 0 \\
\Leftrightarrow\ & \tan\alpha = \frac{1000 \pm \sqrt{1000^2 - 11025}}{245} \\
\Leftrightarrow\ & \tan\alpha = 0.02256235986 \text{ or } 8.140702946 \\
\Rightarrow\ & \alpha \approx 1.3^\circ \text{ or } 83^\circ
\end{aligned}$$

Either choice of $\alpha$ will do.

7. [avg: 5.6/12] Plot the two curves with polar equations

$$r_1 = \sin\theta \text{ and } r_2 = \cos(2\theta),$$

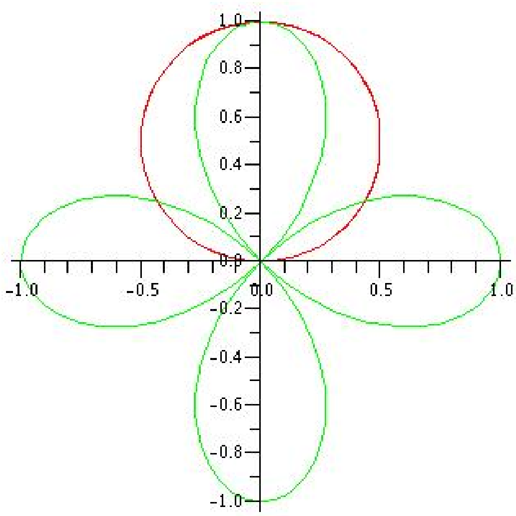
and find the area of the region that is inside the curve $r_2$ but outside the curve $r_1$.

**Solution:** at the intersection,

$$r_1 = r_2 \Rightarrow \sin\theta = \cos(2\theta) \Rightarrow \sin\theta = 1 - 2\sin^2\theta \Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$$

$$\Rightarrow (2\sin\theta - 1)(\sin\theta + 1) = 0 \Rightarrow \sin\theta = \frac{1}{2} \text{ or } \sin\theta = -1.$$

The solutions are $\theta = \pi/6, 5\pi/6$ or $\theta = -\pi/2$, only the first of which is really needed.




In the diagram to the left, the red curve is the polar graph of the circle

$$r_1 = \sin\theta$$

and the green curve is the polar graph of the four-leaved rose

$$r_2 = \cos(2\theta).$$

The region outslde the red circle but inside the green curve consists of two complete petals of the four-leaved rose and two parts of the left and right petals of the four-leaved rose. The total area is given by

$$\begin{aligned}
A &= \underbrace{2\left(\frac{1}{2}\int_{-\pi/4}^{\pi/4} r_2^2\, d\theta\right)}_{\text{two complete petals}} + \underbrace{2\left(\frac{1}{2}\int_0^{\pi/6} \left(r_2^2 - r_1^2\right)\, d\theta\right)}_{\text{bits in two petals above } x\text{-axis, outside circle}} \\
&= 2\int_0^{\pi/4} \cos^2(2\theta)\, d\theta + \int_0^{\pi/6} \left(\cos^2(2\theta) - \sin^2\theta\right)\, d\theta \\
&= \int_0^{\pi/4} (1 + \cos(4\theta))\, d\theta + \int_0^{\pi/6} \left(\frac{1+\cos(4\theta)}{2} - \frac{1-\cos(2\theta)}{2}\right)\, d\theta \\
&= \int_0^{\pi/4} (1 + \cos(4\theta))\, d\theta + \int_0^{\pi/6} \left(\frac{\cos(4\theta)}{2} + \frac{\cos(2\theta)}{2}\right)\, d\theta \\
&= \left[\theta + \frac{\sin(4\theta)}{4}\right]_0^{\pi/4} + \left[\frac{\sin(4\theta)}{8} + \frac{\sin(2\theta)}{4}\right]_0^{\pi/6} \\
&= \frac{\pi}{4} + \frac{\sqrt{3}}{16} + \frac{\sqrt{3}}{8} \\
&= \frac{\pi}{4} + \frac{3\sqrt{3}}{16}
\end{aligned}$$

8. [avg: 3.7/10] What is the $14th$ Maclaurin polynomial[1] of $f(x) = e^{-x^2}\arctan(x^4)$?

**Solution:** use the series for $e^z$ with $z = -x^2$ and the series for $\arctan z$ with $z = x^4$, multiply the series out, and then collect all the terms of degree 14 or less:

$$\begin{aligned} & e^{-x^2}\arctan(x^4) \\ = & \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \frac{x^{10}}{5!} + \cdots\right)\left(x^4 - \frac{x^{12}}{3} + \cdots\right) \\ = & x^4 - \frac{x^{12}}{3} - x^6 + \frac{x^{14}}{3} + \frac{x^8}{2} - \frac{x^{16}}{6} - \frac{x^{10}}{6} + \frac{x^{18}}{18} + \frac{x^{12}}{24} - \frac{x^{20}}{72} - \frac{x^{14}}{120} + \cdot \\ = & x^4 - x^6 + \frac{x^8}{2} - \frac{x^{10}}{6} - \frac{7x^{12}}{24} + \frac{13x^{14}}{40} + \cdots \end{aligned}$$

So $14th$ Maclaurin polynomial of $f(x)$ is

$$P_{14}(x) = x^4 - x^6 + \frac{x^8}{2} - \frac{x^{10}}{6} - \frac{7x^{12}}{24} + \frac{13x^{14}}{40}.$$

[1] same as the $14th$ degree Maclaurin polynomial

9. [avg: 7.0/12] Consider the curve with vector equation

$$\mathbf{r} = (t - \sin t)\,\mathbf{i} + (1 - \cos t)\,\mathbf{j} + 2\sqrt{5}\,\cos\left(\frac{t}{2}\right)\,\mathbf{k}$$

for $0 \leq t \leq 2\pi$.

(a) [6 marks] Calculate both $\dfrac{d\mathbf{r}}{dt}$ and $\left\|\dfrac{d\mathbf{r}}{dt}\right\|$.

**Solution:**

$$\frac{d\mathbf{r}}{dt} = (1 - \cos t)\,\mathbf{i} + \sin t\,\mathbf{j} - \sqrt{5}\,\sin\left(\frac{t}{2}\right)\,\mathbf{k};$$

$$\begin{aligned}
\left\|\frac{d\mathbf{r}}{dt}\right\| &= \sqrt{(1 - \cos t)^2 + \sin^2 t + 5\sin^2\left(\frac{t}{2}\right)} \\
&= \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t + 5\sin^2\left(\frac{t}{2}\right)} \\
&= \sqrt{2 - 2\cos t + 5\sin^2\left(\frac{t}{2}\right)} \\
&= \sqrt{4\sin^2\left(\frac{t}{2}\right) + 5\sin^2\left(\frac{t}{2}\right)} \\
&= \sqrt{9\sin^2\left(\frac{t}{2}\right)} \\
&= 3\sin\left(\frac{t}{2}\right), \text{ since } 0 \leq t \leq 2\pi \Rightarrow 0 \leq \frac{t}{2} \leq \pi \Rightarrow \sin\left(\frac{t}{2}\right) \geq 0
\end{aligned}$$

(b) [6 marks] Find an arc length parameterization of the curve, with reference point $(\pi, 2, 0)$, for which $t = \pi$.

**Solution:** for $0 \leq t \leq 2\pi$,

$$s = \int_\pi^t \left\|\frac{d\mathbf{r}}{du}\right\| du = \int_\pi^t 3\sin\left(\frac{u}{2}\right) du = \left[-6\cos\left(\frac{u}{2}\right)\right]_\pi^t = -6\cos\left(\frac{t}{2}\right).$$

Then $t = 2\cos^{-1}\left(-\frac{s}{6}\right)$ and an arc length parametrization of the curve is

$$\mathbf{r} = \left(2\cos^{-1}\left(-\frac{s}{6}\right) - \sin\left(2\cos^{-1}\left(-\frac{s}{6}\right)\right)\right)\mathbf{i} + \left(1 - \cos\left(2\cos^{-1}\left(-\frac{s}{6}\right)\right)\right)\mathbf{j} - \frac{s\sqrt{5}}{3}\mathbf{k}.$$

**Bonus:** this can be simplified to

$$\mathbf{r} = \left(2\cos^{-1}\left(-\frac{s}{6}\right) + \frac{s\sqrt{36 - s^2}}{18}\right)\mathbf{i} + \left(2 - \frac{s^2}{18}\right)\mathbf{j} - \frac{s\sqrt{5}}{3}\mathbf{k},$$

for $-6 \leq s \leq 6$.